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AN INTRODUCTION TO THE APPLICATION OF FEYNMAN PATH INTEGRALS TO--ETC(U)
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An Introduction to the Application of Feynman Path Integrals to Sound Propagation in the Ocean

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We review and unify those applications and techniques associated with Feynman's theory of path integrals which have been found relevant for sound propagation in the ocean. After giving an introductory discussion of functional integrals in general and Feynman path integrals in particular, we derive several path integral representations for the solutions to the two- and three-dimensional parabolic equations. The analogies which exist between sound propagation, the nonrelativistic quantum mechanics of a point particle, and Brownian motion are considered. Next we use the path integral to derive several methods of approximation including perturbation theory, the Rytov approximation, ray acoustics,
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and straight-line geometric optics. The formalism is then applied to the problem of developing algorithms for numerically solving the parabolic equation. After developing path integral representations for the solution to the Helmholtz equation, we give an extensive discussion of the parabolic approximation.

CONTENTS

AN INTRODUCTION TO THE APPLICATION OF FEYNMAN PATH INTEGRALS TO SOUND PROPAGATION IN THE OCEAN

1. INTRODUCTION

In an increasing number of situations, one is interested in determining the features of acoustic propagation in a general, range-dependent, ocean environment. By way of illustration, during a recent meeting of the Acoustical Society of America [1] 55% of the talks devoted to underwater propagation dealt with phenomena which cannot be adequately modeled in terms of an ocean medium possessing range-independent characteristics.

Of the techniques which are available for modeling propagation in a general environment, the parabolic-equation technique of Tappert and Hardin [2] is perhaps the most promising. It has a wider region of validity than ray-tracing techniques, it takes into account mode coupling to all orders, and it can be implemented numerically using relatively simple algorithms.

The technique assumes that the solution to the Helmholtz equation for the pressure field is not so much different from the solution to a simpler equation called the parabolic equation. Because the Helmholtz equation is an elliptic differential equation, one must simultaneously solve for the field at all points in order to obtain the field at any particular point. This requires considerable computation for a range-dependent medium. On the other hand, the structure of the parabolic equation allows one to obtain its solution at a particular range knowing only its solution at shorter ranges. The equation can therefore be solved by marching in the sense that the field at range r can be obtained by propagating the field at a somewhat shorter range $r - \Delta r$ according to a version of Huygens' principle. The computational realization of this concept leads to the above-mentioned algorithms, the best known of which is the split-step Fourier algorithm [2,3].

Most of the past interest in the parabolic equation has centered about this computational technique. The equation was almost always discussed in the context of some scheme for obtaining a numerical solution. It was realized only recently that the equation is also a useful starting point for analytic studies of sound propagation.

Of the analytic tools available for study of the parabolic equation, Feynman's theory of path integrals [4] seems particularly powerful. It is very intuitive. One expresses the full wave solution to the parabolic equation in terms of the quantities associated with ray acoustics. The split-step Fourier algorithm follows directly in a few lines from the discrete version of the path integral. In fact the most efficient way of solving the parabolic equation is by evaluating the path integral. The split-step algorithm is just one of several techniques for doing this. The primary defect of Tappert and Hardin's technique is that a realistic ocean bottom cannot be modeled. This is a fault of the algorithm rather than the basic parabolic approximation. Feynman's theory points the way toward the development of an algorithm for computing the pressure field above and in a physical ocean bottom. The parabolic approximation itself is most

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naturally discussed using the path-integral formalism. Finally, Dashen [5] has recently developed a unified theory of sound propagation through a random ocean by using path integrals.

The purpose of this report is to dispel some of the mystery which surrounds path integrals and their application to sound propagation. We have attempted to present the introductory material which anyone interested in using path integrals as a research tool would need or want to know. In essence the report consists of a collection of examples and applications. There does not seem to be any way of becoming comfortable with path integrals without working through many examples. Although the report is rather mathematical, containing over 500 equations, the mathematics which is used is pedestrian. The whole subject is treated at an elementary level. For those interested in a more sophisticated discussion of Feynman's theory, several excellent review articles are available. Among these we have found the works by Montroll [6], Gel'fand and Yaglom [7], Brush [8], Tarski [9], Kravtsov [10], Brittin and Chappell [11], Fradkin [12], Berry and Mount [13], Klyatskin [14], Keller and McLaughlin [15], and Koeling and Malfliet [16] particularly useful. The standard reference is the monograph by Feynman and Hibbs [17]. In addition to a general exposition of the subject this work contains a host of useful computational techniques.

This report is organized into nine sections and two appendixes. Section 2 contains some comments concerning functional integrals in general and Feynman path integrals in particular. After Feynman path integral representations are derived for the two- and three-dimensional parabolic equations in Section 3, several alternative representations are developed in Section 4. In Section 5 the analogies are pointed out which exist between sound propagation, quantum mechanics, and Brownian motion. In Section 6 path integrals are used to derive several methods of approximation. Boundary conditions are discussed in Section 7, and the utility of path integrals for developing algorithms is emphasized. The solution to the Helmholtz equation is written as a path integral in Section 8, and several applications are considered. Section 9 is a discussion of the parabolic approximation. This report does not specifically consider the application of Feynman's theory to the problem of sound propagation through a random medium, because the subject has been treated with thoroughness by Dashen [5]. However, so that some previous work can be discussed, acoustic fluctuations are briefly considered in Section 9. Appendix A contains the derivation of an intergral representation for the n-dimensional Green's function in an infinite, homogeneous medium. This representation is used throught the report. In Appendix B the stationary-phase approximation of an integral is recalled.

The various sections of the report are not intended to be read separately. Each section uses previously derived results.

2. FUNCTIONAL INTEGRALS

2.1 General Comments

A functional integral is a generalization of an ordinary N-dimensional integral. To obtain the value of the N-dimensional integral

$$I_N = \int dz_1 \ \ldots \ dz_N \, J(z_1, \ldots, z_N), \tag{2.1}$$

one evaluates the integrand J over the range of the N variables $z_1, \ldots, z_N$. These values of J are then added according to the rules of calculus to determine I_N. A functional integral corresponds to the continuum limit of I_N. The index i of z_i becomes a continuous variable, say r.

$$\{z_1, z_2, \ldots, z_N\} \to \{z(r_1), z(r_2), \ldots, z(r_N)\} \to z(r), \quad r \text{ continuous.} \tag{2.2a}$$

The integral over the N-tuple $\{z_1, \ldots, z_N\}$ becomes an integral over the function $z(r)$:

$$\int dz_1 \cdots dz_N \to \int D[z(r)]. \tag{2.2b}$$

And the integrand J becomes a functional, that is a function of a function:

$$J(z_1, \ldots, z_N) \to J(z(r)). \tag{2.2c}$$

One writes for the functional integral

$$I = \int D[z(r)]\, J(z(r)). \tag{2.3}$$

Operationally it means J is evaluated for all permitted functio (r) and then these contributions are added according to the rules of *functional* calculus to g of the integral.

There are three important points concerning functional integ irst point is that $J(z(r))$ is not simply a function of the parameter r, it cannot be determ y picking a value of r, evaluating z at that point, and then determining J for that value of z. The functional J can be determined only if the complete function $z(r)$ is specified, not its value at any one point. An example of a functional which illustrates this point is the area $A(z(r))$ under a curve $z(r)$ as shown in Fig. 1. Obviously A has a value which cannot be determined unless the complete function z is given. Knowing z at some particular value of r is not enough. It should be clear from this example that $J(z(r))$ is the same functional as $J(z(s))$ and

$$\int D[z(r)]\, J(z(r)) = \int D[z(s)]\, J(z(s)).$$

Our notation, though traditional, is somewhat misleading in this regard. In the mathematics literature one often finds the preferable notation $J(z(\cdot))$ and $\int D[z(\cdot)]\, J(z(\cdot))$.

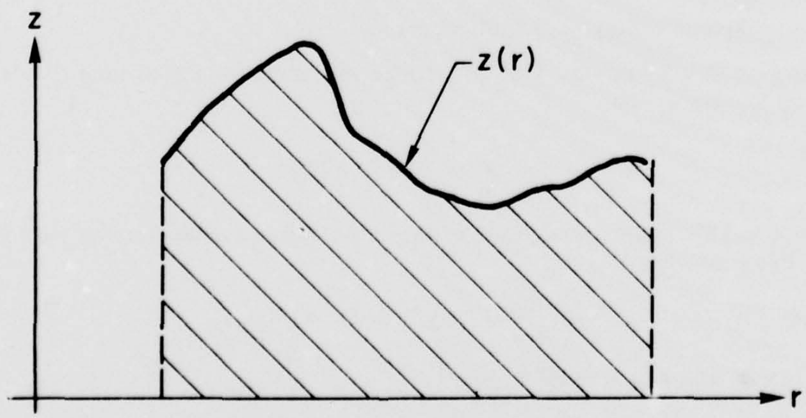

Fig. 1 — The area under a curve as an example of a functional;
$A(z(r)) = \int dr\, z(r)$.

The second point is that one must know the class of functions involved to determine the value of a functional integral. This corresponds to the fact that the N-dimensional integral cannot be evaluated unless the limits of integration are given. The particular specification of the class of functions depends on the problem, but some specification must always be given. As an example, one might specify that the integration is over all continuous and, of course, single-valued functions $z(r)$ defined on the interval $\alpha \leqslant r \leqslant \beta$ such that $z(\alpha) = z_1$, $z(\beta) = z_2$, and $A \leqslant z(r) \leqslant B$.

The third point concerns how the various contributions are added to form the integral. Integral calculus is just the study of the rules by which this addition is accomplished. Different sets of rules correspond to different types of integration, e.g., Riemann-Stieltjes, Lebesque, etc. These rules determine how the contributions to the integral are to be weighted; they determine the *measure*. The measure must always be specified. An expression such as Eq. (2.3) is meaningless unless the measure is explicitly or implicity indicated. Without it one would have no idea how to go about determining the integral's value.

2.2 Feynman Path Integrals

In 1948 Feynman [4] published a formulation of nonrelativistic quantum mechanics conceptually distinct from the formulations developed during the mid-1920's by Schrödinger and Heisenberg. In Feynman's formulation the wave equation and operator calculus are replaced by a type of functional integration called *path integration*.

Specifically, consider the one dimensional motion of a particle of mass m moving under the influence of a position- and time-dependent force $F(t, z)$. If this force is derivable from a potential,

$$F(t, z) = -\partial_z V(t, z), \tag{2.4}$$

then the Lagrangian for the system is

$$\begin{aligned} L &= \frac{1}{2} m \left(\frac{dz}{dt} \right)^2 - V(t, z) \\ &= \text{kinetic energy} - \text{potential energy}. \end{aligned} \tag{2.5}$$

According to Hamilton's Principle [18] the particle will move from z_a at time t_a to z_b at time t_b in such a way that the action

$$A = \int_{t_a}^{t_b} dt\, L(z, t) \tag{2.6}$$

is an extremum. This requirement leads to the differential equation for the path followed by the particle (Newton's law):

$$m \frac{d^2 z}{dt^2} = -\partial_z V(t, z) = F(t, z), \tag{2.7}$$

where $z(t_a) = z_a$ and $z(t_b) = z_b$.

If now we consider the particle to be described by quantum-mechanical rather than classical dynamics, its motion is characterized by a probability amplitude $\Psi(t_a, z_a | t_b, z_b)$. The

modulus of Ψ gives the probability that the particle will travel from z_a at time t_a to z_b at time t_b. Feynman showed that

$$\Psi = \sum_{paths} e^{iA/\hbar}, \tag{2.8}$$

where A is given by Eq. (2.6), $\hbar$ is Planck's constant divided by 2π, and the sum is over all the possible paths, not just the classical one(s), connecting (t_a, z_a) to (t_b, z_b). Equation (2.8) may be rewritten as

$$\Psi(t_a, z_a | t_b, z_b) = \int D[z(t)] \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left[\frac{1}{2} m \left(\frac{dz}{dt} \right)^2 - V(t, z(t)) \right] \right\}. \tag{2.9}$$

The complete quantum-mechanical behavior of the particle is summarized by Eq. (2.9). Although no mention need be made of the differential equation satisifed by Ψ or of the commutation relations satisfied by the observables, Feynman demonstrated that his formulation is mathematically equivalent to the traditional formulations.

Because of its close connection to classical physics, Feynman's theory is perhaps the most intuitive approach to quantum mechanics. It has become increasingly important in theoretical physics. For example, since 1965 an average of at least one paper every nine days has been published in which some aspect of Feynman's theory is considered. Applications have been found in quantum field theory [7,9,12,19-22], statistical mechanics [6-8,11,12,17,23-27], the development of asymptotic expansions for solutions of differential equations [13,15,28-29], the development of variational techniques [17,30], the formulation of semiclassical approximations to scattering processes [16,31-33], the determination of bound states [34], the quantization of field equations [35-39], the study of disordered systems [40-47], and the somewhat related problem of wave propagation through a random medium [5,10,14,48,49]. The theory has generated interest among mathematicians in the problem of rigorously defining path integrals [50-55] and in the relationship between functional integrals and differential equations [7,56-59]. Moreover work has been devoted to approximation schemes and the numerical evalution of path integrals [8,60-69]. This incomplete list gives some idea of the impact the theory has had on physics and mathematics.

2.3 The Question of Rigor

Feynman path integrals have never been given a rigorous mathematical definition. The basic problem is that the functional integral of Eq. (2.9) does not have a countably additive measure [50]. This statement simply means that, with the usual methods of measure theory, path integrals are senseless. Although there exist more-or-less satisfactory definitions of path integrals [50-55], they apply only to certain forms of the potential $V(t, z)$.

How should an acoustician view this lack of mathematical rigor? He should probably ignore it and adopt the point of view that the path integral "if manipulated in a purely formal style without any regard for rigorous justification, gives all the right answers" [70]. After all, if one *does* find an inconsistency which is solely the result of the use of path integrals, he has made a discovery of profound importance.

3. THE FEYNMAN PATH-INTEGRAL SOLUTION TO THE PARABOLIC EQUATION

3.1. Some Preliminaries

Throughout this survey we assume one is interested in determining the pressure field $p(\mathbf{x}, t)$ at some point $\mathbf{x} = (x, y, z)$ in the ocean and at some time t due to a point source located at $\mathbf{x}_s = (x_s, y_s, z_s)$. One traditionally assumes p obeys the wave equation

$$\left[\nabla^2 - \frac{1}{c^2(\mathbf{x}, t)} \partial_t^2\right] p(\mathbf{x},t) = -\delta^{(3)}(\mathbf{x} - \mathbf{x}_s) f(t), \tag{3.1}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$. The function $f(t)$ defines the spectral characteristics of the source. (For a monochromatic source, $f(t) = a\cos(\omega_o t + \phi)$.) The temporal variability of the sound speed $c(\mathbf{x}, \mathrm{t})$ is usually gradual enough so that Eq. (3.1) can be replaced by the Helmholtz equation

$$\left\{\nabla^2 + \left[\frac{\omega}{c(\mathbf{x}, t)}\right]^2\right\} p_\omega(\mathbf{x}, t) = -\delta^{(3)}(\mathbf{x} - \mathbf{x}_s), \tag{3.2}$$

where

$$p(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\omega\, g(\omega) p_\omega(\mathbf{x}, t)\, e^{-i\omega t}, \tag{3.3a}$$

with

$$f(t) = \int_{-\infty}^{\infty} d\omega\, g(\omega) e^{-i\omega t}. \tag{3.3b}$$

In addition to satisfying the Helmholtz equation, $p_\omega(\mathbf{x}, \mathrm{t})$ will satisfy the appropriate boundary conditions at the top and bottom of the ocean and will represent outgoing radiation. With regard to the solution of Eq. (3.2), both ω and t enter as free parameters, i.e., the equation is solved as if they were constant. We shall therefore suppress the dependence of p on these variables and write $p(\mathbf{x})$ for the complex pressure $p_\omega(\mathbf{x}, t)$.

In the parabolic approximation one assumes

$$p(\mathbf{x}) \simeq \text{const}\, \frac{1}{R^{1/2}}\, e^{ik_o R}\, \psi(R, z), \tag{3.4}$$

where $R = \sqrt{(x - x_s)^2 + (y - y_s)^2}$ and $k_o = \omega/c_o$, with c_o being some reference sound speed. The reduced field ψ satisfies the parabolic equation

$$-2ik_o\, \partial_r\, \psi(r, z) = \{\partial_z^2 + k_o^2 [n^2(r, z) - 1]\}\, \psi(r, z), \tag{3.5}$$

where $n(r, z) = c_o/c(r, z)$, with

$$c(r, z) \equiv c\left[x_s + \frac{r}{R}(x - x_s), y_s + \frac{r}{R}(y - y_s), z, t\right]. \tag{3.6}$$

The variable r is the horizontal path length in the direction of propagation. (It is commonly said that the parabolic approximation is related somehow to cylindrical symmetry. Strictly speaking this is not correct. What the parabolic approximation implies is that the acoustic energy is restricted to the vertical slice of the ocean containing the source and receiver. Since only this slice is important, the three-dimensional problem reduces to a problem involving only two coordinate variables: r and z.)

The solution to Eq. (3.5) satisfies the same boundary conditions at the top and bottom of the ocean satisfied by $p(\mathbf{x})$ and an initial condition. This initial condition is usually specified by requiring that ψ equal some depth-dependent function at a small horizontal distance ϵ from the source. That is,

$$\psi(\epsilon, z) = h(z). \tag{3.7}$$

The precise form of h and the value of ϵ depend on the constant coefficient in Eq. (3.4), the acoustic properties of the ocean in the region of the source, and one's inclination. For any choice of initial field, however, ψ can be written in terms of a Green's function Ψ:

$$\psi(r, z) = \int dz_0 \, \Psi(r, z|\epsilon, z_0)\, h(z_0), \qquad r \geqslant \epsilon, \tag{3.8}$$

where

$$\{+2ik_o\partial_r + \partial_z^2 + k_o^2\,[n^2(r, z) - 1]\}\, \Psi(r, z|r', z') = 0 \tag{3.9a}$$

and, for any value of r',

$$\Psi(r', z|r', z') = \delta(z - z'). \tag{3.9b}$$

This Green's function propagates the acoustic energy from a source at r', z' to the point r, z. We always take $r > r'$ in Eq. (3.9a).

If the effects of the variation of the sound speed in cross range are considered important, one could use the three-dimensional version of the parabolic equation. Taking the direction of propagation to be along the positive x axis, we have

$$p(\mathbf{x}) \simeq \text{const}\; e^{ik_o(x - x_s)}\, \psi(x - x_s, \hat{\rho}), \tag{3.10}$$

where $\hat{\rho}$ is a two-dimensional vector, $\hat{\rho} = (y, z)$,

$$\psi(r, \hat{\rho}) = \int^{\infty} d\hat{\rho}_o \, \Psi(r, \hat{\rho}|\epsilon, \hat{\rho}_o)\, h(\hat{\rho}_o), \tag{3.11}$$

$$\{+2ik_o\, \partial_r + \hat{\nabla}^2 + k_o^2\,[n^2(r, \hat{\rho}) - 1]\}\, \Psi(r, \hat{\rho}|r', \hat{\rho}') = 0, \tag{3.12a}$$

and

$$\Psi(r', \hat{\rho}|r', \hat{\rho}') = \delta^{(2)}(\hat{\rho} - \hat{\rho}'), \tag{3.12b}$$

with $\hat{\nabla}^2 = \partial_y^2 + \partial_z^2$ and $n(r, \hat{\rho}) = c_o/c(x_s + r, y, z, t)$.

3.2 The Huygens-Fresnel Principle

Of the many methods available for developing the Feynman path-integral solution to the parabolic equation, perhaps the easiest to understand is the one based on the Huygens-Fresnel Principle [4]. We will illustrate this method by considering the solution to Eq. (3.5) in an unbounded medium.

The Huygens-Fresnel principle gives a technique for constructing the field at r in terms of the field at a shorter range r', where $r - r' = \Delta r$ is taken to be small. Every element of the field at r' is considered the source of secondary wavelets which coherently sum to produce the field at r. These wavelets are solutions to the governing wave equation with constant index of refraction. That is, the rays associated with the wavelets follow straight paths. Let z_r be a depth variable at range r and let $z_{r'}$ be a depth variable at r' as illustrated in Fig. 2. (It might

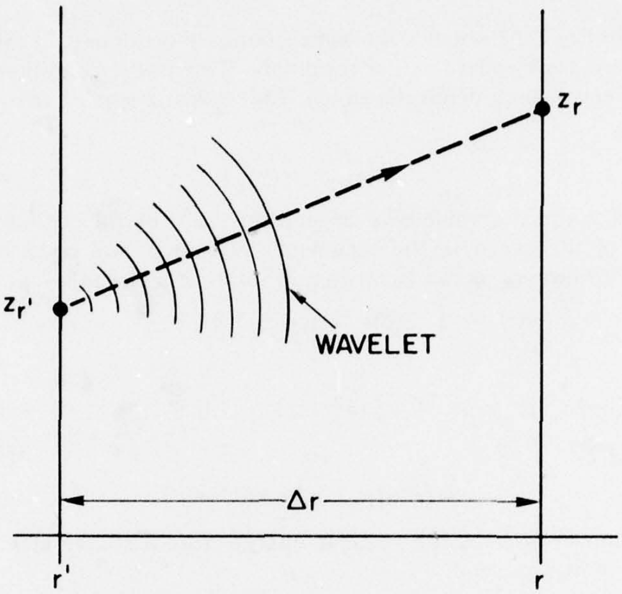

Fig. 2 — The Huygens-Fresnel Principle

be more customary to simply use z and z' for z_r and $z_{r'}$ respectively. We chose this notation because it will suggest a functional integration.) According to the Huygens-Fresnel principle [71] the contribution to $\psi(r, z_r)$ from an element at $r', z_{r'}$ is

$$d\psi(r, z_r) = \eta(r', z_{r'})\, w(r, z_r\,;\, r', z_{r'})\psi(r', z_{r'})dz_{r'}, \tag{3.13}$$

where $w(r, z_r\,;\, r', z_{r'})$ is the field associated with the wavelet. The presence of the phenomenological factor η is characteristic of the Huygens-Fresnel theory; it is usually defined within the context of the Kirchhoff formulation. It will be determined below by a consistency condition.

Now $w(r, z_r\,;\, r', z_{r'})$ satisfies Eqs. (3.9) with $n^2(r, z_r)$ taken to be constant. Writing

$$w = \exp\left\{\frac{i(r-r')k_o}{2}\,[n^2(r,z_r) - 1]\right\}\overline{w}, \tag{3.14a}$$

we have

$$(+2ik_o\partial_r + \partial_{z_r}^2)\overline{w} = 0 \tag{3.14b}$$

and

$$\overline{w}\big|_{r=r'} = \delta(z_r - z_{r'}). \tag{3.14c}$$

Eqs. (3.14) are easily solved by Fourier transforming with respect to z_r. We obtain

$$w(r, z_r\,;\, r', z_{r'}) = \left[\frac{k_o}{2\pi i(r - r')}\right]^{1/2} \exp\left\{\frac{ik_o(r - r')}{2}\left[\left[\frac{z_r - z_{r'}}{r - r'}\right]^2 + n^2(r, z_r) - 1\right]\right\}. \tag{3.15}$$

Integrating Eq. (3.13) over all the elements gives

$$\psi(r, z_r) = \left[\frac{k_o}{2\pi i (r - r')}\right]^{1/2} \int_{-\infty}^{\infty} dz_{r'} \exp\left\{\frac{ik_o(r-r')}{2}\left[\left(\frac{z_r - z_{r'}}{r - r'}\right)^2 + n^2(r, z_r) - 1\right]\right\} \times \eta(r', z_{r'}) \psi(r', z_{r'}). \tag{3.16}$$

The factor η is determined by the condition that Eq. (3.16) reduce to an identity as $r \to r'$. In this limit

$$\psi(r, z_r) \to \int dz_{r'}\, \eta(r, z_{r'})\, \delta(z_r - z_{r'})\, \psi(r, z_{r'}) \to \eta(r, z_r)\, \psi(r, z_r),$$

implying

$$\eta = 1. \tag{3.17}$$

(Equation (3.17) is also obtained from the parabolic-equation analog of the Kirchhoff theory.)

It is convenient to define

$$U(r, z) = -\frac{1}{2}[n^2(r, z) - 1]. \tag{3.18}$$

If $n(r, z) = 1 - \kappa(r, z)$, where κ is a small perturbation, then $U(r, z) \simeq \kappa(r, z)$.

In summary, the field at r is given in terms of the field at r' by

$$\psi(r, z_r) = \left[\frac{k_o}{2\pi i \Delta r}\right]^{1/2} \int_{-\infty}^{\infty} dz_{r'} \exp\left\{ik_o\Delta r\left[\frac{1}{2}\left(\frac{z_r - z_{r'}}{\Delta r}\right)^2 - U(r, z_r)\right]\right\} \psi(r', z_{r'}), \tag{3.19}$$

where $\Delta r = r - r'$.

By carrying out the indicated differentiations, one can show

$$\begin{aligned}\{-2ik_o\partial_r - \partial_{z_r}^2 - k_o^2[n^2(r, z_r) - 1]\}\,\psi(r, z_r) &= \left(\frac{ik_o\Delta r}{2}\right)\left\{\left[-2ik_o\,\partial_r n^2(r, z_r) - \partial_{z_r}^2 n^2(r, z_r) \right.\right.\\ &\left.\left. - \frac{ik_o\Delta r}{2}[\partial_{z_r} n^2(r, z_r)]^2\right]\psi(r, z_r) - 2\partial_{z_r} n^2(r, z_r)\partial_{z_r}\psi(r, z_r)\right\}.\end{aligned} \tag{3.20}$$

If Δr is chosen small enough so that the gradients of $n^2(r, z_r)$ can be ignored, the right-hand side of Eq. (3.20) is insignificant and Eq. (3.19) is a good approximation to the solution of Eq. (3.5). In other words, if Δr is picked small enough so that the rays associated with each wavelet do not get a chance to significantly bend as a result of the variation of the index of refraction, Eq. (3.19) will give a valid approximation to the field at r and z_r.

By an iterative procedure we will now obtain the field at range $r = R$ and depth z due to a source at $r = 0$ with the spatial distribution at $r = \epsilon$ given by Eq. (3.7). We partition the plane of propagation into N strips of width Δr by lines at $r = r_1, r_2, \ldots, r_{N-1}$ as indicated in Fig. 3. Just as before, a depth coordinate z_{r_i} is associated with the boundary $r = r_i$. The notation is simplified if we set

$$r_0 = \epsilon,\ z_{r_0} = z_0,\ r_N = R,\ z_{r_N} = z. \tag{3.21}$$

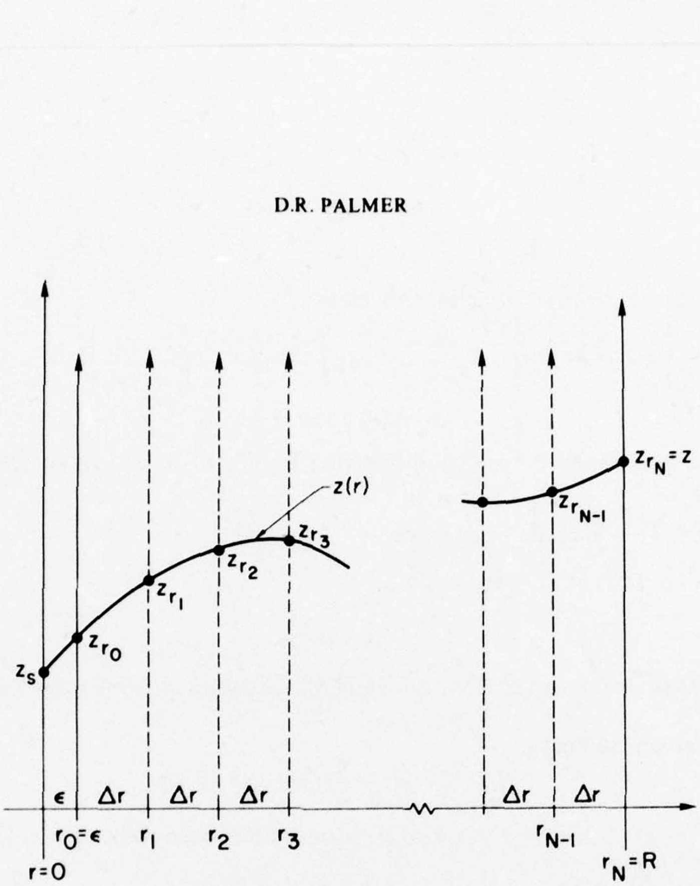

Fig. 3 — A single Feynman path $z(r)$. The summation over all the paths amounts to integrating over $z_{r1}, z_{r2}, ..., z_{rN-1}$.

The Huygens-Fresnel principle will now be applied to each strip. At $r = r_0$ we have, from Eq. (3.7),

$$\psi(r_0, z_{r_0}) = h(z_{r_0}). \tag{3.22a}$$

At the end of the first strip

$$\psi(r_1, z_{r_1}) = \left[\frac{k_o}{2\pi i \Delta r}\right]^{1/2} \int dz_{r_0}\, h(z_{r_0}) \times \exp\left\{ik_o\Delta r\left[\frac{1}{2}\left(\frac{z_{r_1} - z_{r_0}}{\Delta r}\right)^2 - U(r_1, z_{r_1})\right]\right\}. \tag{3.22b}$$

At the end of the second strip

$$\begin{aligned}\psi(r_2, z_{r_2}) &= \left[\frac{k_o}{2\pi i \Delta r}\right]^{1/2} \int dz_{r_1}\, \psi(r_1, z_{r_1}) \times \exp\left\{ik_o\Delta r\left[\frac{1}{2}\left(\frac{z_{r_2} - z_{r_1}}{\Delta r}\right)^2 - U(r_2, z_{r_2})\right]\right\} \\ &= \left[\frac{k_o}{2\pi i \Delta r}\right]^{2/2} \int dz_{r_0}\, dz_{r_1}\, h(z_{r_0}) \times \exp\left\{ik_o\Delta r \sum_{i=1}^{2}\left[\frac{1}{2}\left(\frac{z_{r_i} - z_{r_{i-1}}}{\Delta r}\right)^2 - U(r_i, z_{r_i})\right]\right\}.\end{aligned} \tag{3.22c}$$

At the end of N strips

$$\psi(r_N, z_{r_N}) \equiv \psi(R, z) = \left(\frac{k_o}{2\pi i \Delta r}\right)^{N/2} \int dz_{r_0}\, dz_{r_1} \cdots dz_{r_{N-1}}\, h(z_{r_0})$$

$$\times \exp\left\{ik_o \Delta r \sum_{i=1}^{N}\left[\frac{1}{2}\left(\frac{z_{r_i} - z_{r_{i-1}}}{\Delta r}\right)^2 - U(r_i, z_{r_i})\right]\right\}. \qquad (3.22\text{d})$$

Because of the finite size of Δr, Eq. (3.22d) is only an approximation. The exact solution is obtained as $\Delta r \to 0$. Since $N\Delta r = R - \epsilon$ is fixed, as $\Delta r \to 0$ $N \to \infty$ and the N-dimensional integral in Eq. (3.22d) becomes infinite dimensional. This infinite-dimensional integral is interpreted as a functional or path integral. The points (r_0, z_{r_0}), (r_1, z_{r_1}) ..., (r_N, z_{r_N}) are considered to lie on a continuous curve $z(r)$ such that $z_{r_i} = z(r_i)$, giving $z(\epsilon) = z(r_0) = z_0$ and $z(r_N) = z(R) = z$ (Fig. 3). The $N - 1$ intermediate integrations over z_{r_1}, ..., $z_{r_{N-1}}$ become an integration over $z(r)$:

$$\left(\frac{k_o}{2\pi i \Delta r}\right)^{N/2} \int dz_{r_1} \ldots dz_{r_{N-1}} \to \int D[z(r)]. \qquad (3.23)$$

This equation defines the measure associated with the path integral. The expression $(z_{r_i} - z_{r_{i-1}})/\Delta r = (z_{r_i} - z_{r_{i-1}})/(r_i - r_{i-1})$ is the discrete approximation to the derivative of $z(r)$, and $\Delta r \sum_{i=1}^{N} (\ldots)$ is the discrete approximation to an integral over r. Therefore, as $\Delta r \to 0$,

$$\Delta r \sum_{i=1}^{N}\left[\frac{1}{2}\left(\frac{z_{r_i} - z_{r_{i-1}}}{\Delta r}\right)^2 - U(r_i, z_{r_i})\right] \to \int_\epsilon^R dr\left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 - U(r, z(r))\right]. \qquad (3.24)$$

Collecting expressions gives

$$\psi(R, z) = \int dz_0\, h(z_0) \int D[z(r)] \exp\left\{ik_o \int_\epsilon^R dr\left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 - U(r, z(r))\right]\right\}, \qquad (3.25)$$

where the integration is over all continuous functions $z(r)$ such that $z(\epsilon) = z_0$ and $z(R) = z$. Comparing Eq. (3.25) with Eq. (3.8) gives

$$\Psi(R, z|\epsilon, z_0) = \int D[z(r)] \exp\left\{ik_o \int_\epsilon^R dr\left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 - U(r, z(r))\right]\right\} \qquad (3.26)$$

for the Green's function. For an arbitrary coordinate dependence

$$\Psi(r, z|r', z') = \int D[z(s)] \exp\left\{ik_o \int_{r'}^{r} ds\left[\frac{1}{2}\left(\frac{dz}{ds}\right)^2 - U(s, z(s))\right]\right\}, \qquad (3.27)$$

where $z(s)$ is a continuous curve defined on the interval $r' \leqslant s \leqslant r$ and satisfying the end-point conditions $z(r') = z'$ and $z(r) = z$.

Equations (3.25), (3.26), and (3.27) are exact; we have "only" assumed the path integral exists.

3.3 The Composition Law

In this subsection we will outline a second technique for constructing the path integral. We will work with the Green's function defined by Eqs. (3.9). It satisfies the *composition law*:

$$\Psi(r, z|r', z') = \int dz_s\, \Psi(r, z|s, z_s)\, \Psi(s, z_s|r', z'), \tag{3.28}$$

where

$$r \geqslant s \geqslant r'. \tag{3.29}$$

In the context of the study of Markov processes, Eq. (3.28) is sometimes referred to as the Smoluchowski-Kolmogorov relation.

To show that the composition law is valid, we must show that the right-hand side of Eq. (3.28) satisfies Eqs. (3.9). Equation (3.9a) follows easily enough by operating on both sides of Eq. (3.28) with $2ik_o\partial_r + \partial_z^2 + k_o^2[n^2(r, z) - 1]$ and noting that this operator can be moved inside the integral over z_s. To show Eq. (3.9b) holds, one must take the limit $r \to r'$ without violating Eq. (3.29). This can be done in two ways: $r \to s$ followed by $s \to r'$ or $s \to r'$ followed by $r \to s$. In either case Eq. (3.9b) will be satisfied. This two-step limiting process is artificial and can be avoided by doing the integral over z_s for $r - s$ and $s - r'$ small but finite and then taking the limit $r \to r'$. Below we will obtain an expression for Ψ for small horizontal separations which permits this procedure to be carried out. The result is again Eq. (3.9b).

It is rather interesting that the right-hand side of the composition law is independent of s. This characteristic is a result of the falloff of Ψ at infinity. To see this, we differentiate the right-hand side of Eq. (3.28) with respect to s and use the equation reciprocal to Eq. (3.9a),

$$\{+\,2ik_o\partial_{r'} - \partial_{z'}^2 - k_o^2[n^2(r', z') - 1]\}\,\Psi(r, z|r', z') = 0, \tag{3.30}$$

to obtain

$$\begin{aligned}
\partial_s \int dz_s\, &\Psi(r, z|s, z_s)\, \Psi(s, z_s|r', z')\\
&= \frac{1}{2ik_o}\int dz_s\, [\partial_{z_s}^2\, \Psi(r, z|s, z_s)\, \Psi(s, z_s|r', z')\\
&\quad - \Psi(r, z|s, z_s)\partial_{z_s}^2\, \Psi(s, z_s|r', z')]\\
&= \frac{1}{2ik_o}[\partial_{z_s}\, \Psi(r, z|s, z_s)\, \Psi(s, z_s|r', z')\\
&\quad - \Psi(r, z|s, z_s)\, \partial_{z_s}\, \Psi(s, z_s|r', z')]_{z_s=-\infty}^{z_s=+\infty}\\
&= 0,
\end{aligned} \tag{3.31}$$

provided the Green's function falls off at infinity (or provided the Wronskian-like quantity in the last pair of brackets is independent of z_s).

We now iterate the composition law

$$\Psi(r, z|r', z') = \int dz_1 \ldots dz_{N-1} \prod_{i=1}^{N} \Psi(s_i, z_i|s_{i-1}, z_{i-1}), \tag{3.32}$$

where

$$s_N = r, \quad z_N = z, s_0 = r', \quad z_0 = z'. \tag{3.33}$$

We pick s_i ($i = 1, ..., N - 1$) so that $s_i - s_{i-1} = \Delta s$, giving $N\Delta s = r - r'$. Consider one of the factors in Eq. (3.32). Since $\Psi(s_i, z_i | s_{i-1}, z_{i-1}) = \Psi(s_i, z_i | s_i - \Delta s, z_{i-1})$ is a δ-function in $z_i - z_{i-1}$ for $\Delta s = 0$, it will be "almost like a δ-function" for Δs small. This means that only a small region around $z_i = z_{i-1}$ contributes to the integral in Eq. (3.32). Therefore, in calculating $\Psi(s_i, z_i | s_{i-1}, z_{i-1})$, we need only consider the variation of the index of refraction within a small region about the point s_i, z_i. In particular, by appropriate choice of Δs, this region can be made so small that $n(r, z)$ will be essentially constant within it and

$$\Psi(s_i, z_i | s_{i-1}, z_{i-1}) = \left(\frac{k_o}{2\pi i \Delta s}\right)^{1/2} \exp\left\{\frac{ik_o}{2}\Delta s\left[\left(\frac{z_i - z_{i-1}}{\Delta s}\right)^2 + n^2(s_i, z_i) - 1\right]\right\}, \tag{3.34}$$

which solves Eq. (3.9) for a constant index of refraction. Substituting Eq. (3.34) into Eq. (3.32) and using Eq. (3.18), we obtain

$$\Psi(r, z | r', z') = \left(\frac{k_o}{2\pi i \Delta s}\right)^{N/2} \int dz_1 \ldots dz_{N-1}$$

$$\times \exp\left\{ik_o \Delta s \sum_{i=1}^{N}\left[\frac{1}{2}\left(\frac{z_i - z_{i-1}}{\Delta s}\right)^2 - U(s_i, z_i)\right]\right\} \tag{3.35}$$

for Δs small. Just as in the treatment following Eq. (3.22d), Eq. (3.35) reduces to a functional integral as $\Delta s \rightarrow 0$:

$$\{z_0, z_1, ..., z_N\} \rightarrow z(s), \tag{3.36a}$$

where, from Eq. (3.33), $r' \leqslant s \leqslant r$, $z(r') = z'$, and $z(r) = z$,

$$\left(\frac{k_o}{2\pi i \Delta s}\right)^{N/2} \int dz_1 \ldots dz_{N-1} \rightarrow \int D[z(s)], \tag{3.36b}$$

$$\Delta s \sum_{i=1}^{N}\left[\frac{1}{2}\left(\frac{z_i - z_{i-1}}{\Delta s}\right)^2 - U(s_i, z_i)\right] \rightarrow \int_{r'}^{r} ds\left[\frac{1}{2}\left(\frac{dz}{ds}\right)^2 - U(s, z(s))\right], \tag{3.36c}$$

giving Eq. (3.27) again:

$$\Psi(r, z | r', z') = \int D[z(s)] \exp\left\{ik_o \int_{r'}^{r} ds\left[\frac{1}{2}\left(\frac{dz}{ds}\right)^2 - U(s, z(s))\right]\right\}. \tag{3.27}$$

3.4 The Three-Dimensional Parabolic Equation

The derivation of the path-integral solution to Eqs. (3.12) follows step by step the derivation for the two-dimensional problem. One obtains

$$\Psi(r, \hat{\rho} | r', \hat{\rho}') = \int D[\hat{\rho}(s)] \exp\left\{ik_o \int_{r'}^{r} ds\left[\frac{1}{2}\left(\frac{d\hat{\rho}}{ds}\right)^2 - U(s, \hat{\rho}(s))\right]\right\}. \tag{3.37}$$

The integration is over a two-dimensional vector function $\hat{\rho}(s)$ such that $\hat{\rho}(r') = \hat{\rho}'$ and $\hat{\rho}(r) = \hat{\rho}$. The measure is defined by the rule

$$\left(\frac{k_o}{2\pi i \Delta s}\right)^N \int d\hat{\rho}_1, \ldots, d\hat{\rho}_{N-1} = \left(\frac{k_o}{2\pi i \Delta s}\right)^N \int dy_1 \, dz_1, \ldots, dy_{N-1} \, dz_{N-1}$$
$$\rightarrow \int D[\hat{\rho}(s)]. \tag{3.38}$$

4. ALTERNATIVE REPRESENTATIONS

In the previous section we derived a path-integral solution of the form

$$\int D[z(s)] e^{ik_o A}, \tag{4.1}$$

where A is a functional of the path $z(s)$ and its derivative. This representation is usually called the coordinate representation. For many problems it is not particularly convenient. In this section we develop two alternative representations.

4.1. The Wavenumber Representation

In the wavenumber or momentum representation the number of degrees of freedom is doubled by introducing an additional integration over a function $k(s)$. Physically $k(s)$ represents the vertical wavenumber associated with the path $z(s)$.

We first rewrite Eq. (3.34) in the form

$$\Psi(s_i, z_i | s_{i-1}, z_{i-1}) = k_o \int_{-\infty}^{\infty} \frac{dk_i}{(2\pi)} \exp\left\{ik_o \Delta s \left[k_i \left(\frac{z_i - z_{i-1}}{\Delta s}\right)\right.\right.$$
$$\left.\left. - \frac{1}{2} k_i^2 - U(s_i, z_i)\right]\right\}. \tag{4.2}$$

(We have defined k_i to be dimensionless.) Substituting Eq. (4.2) into Eq. (3.32) yields

$$\Psi(r, z | r', z') = \left(\frac{k_o}{2\pi}\right)^N \int dz_1 \ldots dz_{N-1} \int dk_1 \ldots dk_N \exp\left\{ik_o \Delta s\right.$$
$$\left. \times \sum_{i=1}^{N} \left[k_i \left(\frac{z_i - z_{i-1}}{\Delta s}\right) - \frac{1}{2} k_i^2 - U(s_i, z_i)\right]\right\}. \tag{4.3}$$

In the continuum limit

$$\{z_0, z_1, \ldots, z_N\} \rightarrow z(s), \tag{4.4a}$$

$$\{k_1, \ldots, k_N\} \rightarrow k(s), \tag{4.4b}$$

$$\Delta s \sum_{i=1}^{N} \left[k_i \left(\frac{z_i - z_{i-1}}{\Delta s}\right) - \frac{1}{2} k_i^2 - U(s_i, z_i)\right]$$
$$\rightarrow \int_{r'}^{r} ds \left[k(s) \frac{dz}{ds} - \frac{1}{2} k^2(s) - U(s, z(s))\right], \tag{4.4c}$$

and

$$\left(\frac{k_o}{2\pi}\right)^N \int dz_1, \ldots, dz_{N-1} \int dk_1 \cdots dk_N \rightarrow \int D[z(s)]D[k(s)], \qquad (4.4d)$$

giving

$$\Psi(r, z|r', z') = \int D[z(s)]D[k(s)] \exp\left\{ik_o \times \int_r^{r'} ds\left[k(s)\frac{dz}{ds} - \frac{1}{2}k^2(s) - U(s, z(s))\right]\right\}. \qquad (4.5)$$

The path $k(s)$ is unconstrained, and, as before, $z(r') = z'$ and $z(r) = z$. From a comparison of Eqs. (4.4d) and (3.36c) we notice a different measure is being used for the functional integral in the wavenumber representation.

In three dimensions the wavenumber representation is

$$\Psi(r, \hat{\rho}|r', \hat{\rho}') = \int D[\hat{\rho}(s)]D[\hat{k}(s)] \exp\left\{ik_o \times \int_r^{r'} ds\left[\hat{k}(s) \cdot \frac{d\hat{\rho}}{ds} - \frac{1}{2}\hat{k}^2(s) - U(s, \hat{\rho}(s))\right]\right\}, \qquad (4.6)$$

where

$$\left(\frac{k_o}{2\pi}\right)^{2N} \int d\hat{\rho}_1 \ldots d\hat{\rho}_{N-1} \int d\hat{k}_1 \ldots d\hat{k}_N \rightarrow \int D[\hat{\rho}(s)]D[\hat{k}(s)]. \qquad (4.7)$$

4.2 The Velocity Representation

Many times it is desirable to transform a path integral by employing a change of variable:

$$z(s) \rightarrow z'(s) = \text{some function of } z(s).$$

We shall introduce this technique by transforming Eq. (3.27) into a representation in which the integration is over the "velocity"

$$v(s) = \frac{dz}{ds}.$$

We first observe that Eq. (3.35) can be written in the form

$$\Psi(r, z|r', z') = \left(\frac{k_o}{2\pi i\Delta s}\right)^{N/2} \int dz_1 \ldots dz_N\, \delta(z - z_N) \exp\left\{ik_o\Delta s \times \sum_{i=1}^{N}\left[\frac{1}{2}\left(\frac{z_i - z_{i-1}}{\Delta s}\right)^2 - U(s_i, z_i)\right]\right\}. \qquad (4.8)$$

We now introduce a change of variable

$$v_i = \frac{z_i - z_{i-1}}{\Delta s}, \quad i = 1, \ldots, N, \qquad (4.9)$$

or, since $z_0 = z'$,

$$z_1 = z' + \Delta s v_1,$$
$$z_2 = z' + \Delta s(v_1 + v_2),$$
$$\ldots$$
$$z_N = z' + \Delta s(v_1 + v_2 + \ldots + v_N). \tag{4.10}$$

The Jacobian of the transformation is simply

$$dz_1 \ldots dz_N = (\Delta s)^N dv_1 \ldots dv_N. \tag{4.11}$$

Substituting this into Eq. (4.8) gives

$$\Psi(r, z|r', z') = \left(\frac{k_o \Delta s}{2\pi i}\right)^{N/2} \int dv_1 \ldots dv_N \, \delta\left(z - z' - \Delta s \sum_{i=1}^{N} v_i\right) \exp\left\{ik_o \Delta s \times \sum_{i=1}^{N}\left[\frac{1}{2} v_i^2 - U\left(s_i, z' + \Delta s \sum_{j=1}^{i} v_j\right)\right]\right\}. \tag{4.12}$$

In the continuum limit

$$\Psi(r, z|r', z') = \int D[v(s)] \, \delta\left(z - z' - \int_{r'}^{r} ds \, v(s)\right) \exp\left\{ik_o \times \int_{r'}^{r} ds\left[\frac{1}{2} v^2(s) - U\left(s, z' + \int_{r'}^{s} ds' \, v(s')\right)\right]\right\}. \tag{4.13}$$

Unlike the previous examples, the integration is over a function $v(s)$ which is not constrained by end-point conditions. The measure is determined by the correspondence

$$\left(\frac{k_o \Delta s}{2\pi i}\right)^{N/2} \int dv_1 \ldots dv_N \rightarrow \int D[v(s)]. \tag{4.14}$$

For the three-dimensional problem

$$\Psi(r, \hat{\rho}|r', \hat{\rho}') = \int D[\hat{v}(s)] \, \delta^{(2)}\left(\hat{\rho} - \hat{\rho}' - \int_{r'}^{r} ds \, \hat{v}(s)\right) \exp\left\{ik_o \times \int_{r'}^{r} ds\left[\frac{1}{2} \hat{v}^2(s) - U\left(s, \hat{\rho}' + \int_{r'}^{s} ds' \, \hat{v}(s')\right)\right]\right\} \tag{4.15}$$

where

$$\left(\frac{k_o \Delta s}{2\pi i}\right)^{N} \int d\hat{v}_1 \ldots d\hat{v}_N \rightarrow \int D[\hat{v}(s)]. \tag{4.16}$$

The velocity representation occurs frequently in the Russian literature.

4.3. Normalization Conventions

We have defined the measure by indicating that the functional integral is the limit of a multidimensional integral as the dimensionality approaches infinity and then specifying the normalization of this multidimensional integral. That is, Eq. (3.36b) indicates how Eq. (3.27) is

to be evaluated. This procedure is usually not followed in research articles. Instead the normalization is implicity specified by stating, for example, that the integral in Eq. (3.27) is defined such that

$$\int D[z(s)] \exp\left[\frac{ik_o}{2}\int_{r'}^{r} ds \left(\frac{dz}{ds}\right)^2\right]$$

$$= \left[\frac{k_o}{2\pi i(r - r')}\right]^{1/2} \exp\left[\frac{ik_o(r - r')}{2}\left(\frac{z - z'}{r - r'}\right)^2\right]. \tag{4.17}$$

We want to show in this subsection that this procedure is entirely equivalent to the one we have been using. First, Eq. (4.17) is certainly valid if we assume the conventions given by Eq. (3.36). By comparing Eqs. (4.17) and (3.27) it is clear the left-hand side of Eq. (4.17) is the solution to Eqs. (3.9) with $n = 1$. This solution can be trivially obtained directly from Eqs. (3.9) and is equal to the right-hand side of Eq. (4.17).

We now want to show that Eq. (4.17) does indeed give the normalization indicated in Eq. (3.36b) by explicity evaluating the path integral in Eq. (4.17) The left-hand side of Eq. (4.17) is first written as

$$\lim_{N\to\infty} A(N) \int dz_1 \ldots dz_{N-1} \exp\left[\frac{ik_o}{2\Delta s}\sum_{i=1}^{N}(z_i - z_{i-1})^2\right], \tag{4.18}$$

where, as before, $z_0 = z'$, $z_N = z$, and $\Delta s = (r - r')/N$. We want to show

$$A(N) = \left[\frac{k_o N}{2\pi i(r - r')}\right]^{N/2}. \tag{4.19}$$

Following the steps which led to Eq. (4.12),

$$\int dz_1 \ldots dz_{N-1} \exp\left[\frac{ik_o}{2\Delta s}\sum_{i=1}^{N}(z_i - z_{i-1})^2\right]$$

$$= (\Delta s)^N \int dv_1 \ldots dv_N\, \delta\left[z - z' - \Delta s \sum_{i=1}^{N} v_i\right] \exp\left[\frac{ik_o}{2\Delta s}\sum_{i=1}^{N} v_i^2\right]$$

$$= (\Delta s)^N \int \frac{dk}{(2\pi)} e^{ik(z - z')} \prod_{i=1}^{N} \int dv_i \exp\left[\frac{ik_o}{2\Delta s} v_i^2 - i\Delta s k v_i\right]$$

$$= \left[\frac{2\pi i \Delta s}{k_o}\right]^{N/2} \int \frac{dk}{(2\pi)} \exp\left[ik(z - z') - i\frac{N\Delta s}{2k_o}k^2\right]$$

$$= \left[\frac{2\pi i(r - r')}{k_o N}\right]^{N/2} \left[\frac{k_o}{2\pi i(r - r')}\right]^{1/2} \exp\left[\frac{ik_o}{2(r - r')}(z - z')^2\right]. \tag{4.20}$$

In writing Eq. (4.20) we have used the important formula

$$\int_{-\infty}^{+\infty} dv\, e^{i\alpha v^2 + i\beta v} = \left(\frac{i\pi}{\alpha}\right)^{1/2} e^{-i\beta^2/4\alpha}. \tag{4.21}$$

Comparison of Eqs. (4.20), (4.18), and (4.17) gives (4.19), the desired result.

We have followed a standard scheme in evaluating the path integral: a change of variable is introduced to diagonalize the exponent, and the integrations are then carried out using the formulas for Gaussian integrals.

5. SOME ANALOGIES

5.1 Quantum Mechanics

The quantum-mechanical behavior of a particle of mass m moving along the z axis in a potential $V(t,z)$ is determined by a wave function $\psi(t,z)$ which satisfies Schrödinger's equation

$$-2i\left[\frac{m}{\hbar}\right]\partial_t\,\psi(t,z) = \left[\partial_z^2 - \frac{2m}{\hbar^2}V(t,z)\right]\psi(t,z), \tag{5.1}$$

where $\hbar$ is Planck's constant divided by 2π. If the wave function is known at some initial time t_0, it will be given at a later time by

$$\psi(t,z) = \int dz_0\; K(t,z|t_0,z_0)\,\psi(t_0,z_0), \tag{5.2}$$

where the Green's function kernel (Feynman propagator) K obeys the Schrödinger's equation and the initial condition

$$K(t_0,z|t_0,z_0) = \delta(z-z_0). \tag{5.3}$$

If one compares Eqs. (5.1), (5.2), and (5.3) with Eqs. (3.5), (3.8), and (3.9), one sees there exists an analogy between the nonrelativistic quantum-mechanical dynamics of a point particle and acoustic propagation in the ocean. The correspondence can be made precise by introducing some reference speed c_o which may be though of as the average speed of the particle. Its value is unimportant, since it will cancel in any quantum-mechanical expression. It is introduced simply to make the dimensions work out. In Table 1 we have constructed a dictionary for passing from quantum mechanics to underwater sound propagation. The meaning of several items in the table will become clear when we consider the ray-acoustics approximation to the parabolic equation.

It may be a mistake to attach great importance to this analogy. One cannot solve the problems of sound propagation in the ocean by simply borrowing quantum-mechanical results. Much work has been devoted to quantum mechanics, however, and techniques and specific calculations which are now being discussed by acousticians first appeared in a quantum-mechanical context and have been available for years.

5.2 Brownian Motion

If a particle constrained to move along the z axis undergoes Brownian motion, the position of the particle as a function of time is described by a probability density $\psi(t,z)$ which satisfies the *diffusion equation*

$$\partial_t\,\psi = \frac{D}{2}\,\partial_z^2\,\psi. \tag{5.4}$$

The *diffusion coefficient* D expresses the physical characteristics of the particle and medium via Einstein's relation.

Table 1: A Quantum Mechanics-Acoustic Propagation Dictionary

Quantum Mechanics	Acoustic Propagation
1. Wave function $\psi(t, z)$	Reduced field $\psi(r, z)$
2. Propagator $K(t, z \mid t_o, z_o)$	Green's function $\Psi(r, z \mid r_o, z_o)$
3. Scaled time $c_o t$	Range r
4. DeBroglie wave number $mc_o/\hbar$	Characteristic acoustic wave number k_o
5. Scaled potential energy $V(t, z)/mc_o^2$	Variable part of the index of refraction $U(r, z) = -\frac{1}{2}[n^2(r, t) - 1]$
6. Scaled action $\frac{1}{\hbar}\int dt\left[\frac{1}{2}m\left(\frac{dz}{dt}\right)^2 - V\right]$	Eikonal $k_o \int dr\left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 - U\right]$
7. Classical particle trajectory $z(t)$	Ray path $z(r)$
8. Scaled speed of the classical particle $\frac{1}{c_o}\frac{dz}{dt}$	Slope of the ray path $\frac{dz}{dr}$
9. Scaled Newton's equation $\frac{1}{mc_o^2}\left(m\frac{d^2z}{dt^2} + \partial_z V\right) = 0$	Eikonal equation $\frac{d^2z}{dr^2} + \partial_z U = 0$

For example, if the particle was at $z = z_0$ at time $t = t_0$, then the probability that the particle will be in the interval z to $z + \Delta z$ at time t is

$$P(t, z)\Delta z = \left(\frac{1}{2\pi D(t - t_0)}\right)^{1/2} \exp\left[-\frac{(t - t_0)}{2D}\left(\frac{z - z_0}{t - t_0}\right)^2\right] \Delta z. \tag{5.5}$$

According to the rules for conditional probabilities, the probability that the particle will be in the intervals z_1 to $z_1 + \Delta z_1$, z_2 to $z_2 + \Delta z_2$, ..., z_N to $z_N + \Delta z_N$ at times $t_1, t_2, ..., t_N$ respectively, where $t_0 \leqslant t_1 \leqslant t_2 \leqslant ... \leqslant t_N$, given that it was at z_0 at time t_0, is

$$\prod_{i=1}^{N}\left\{\left(\frac{1}{2\pi D(t_i - t_{i-1})}\right)^{1/2} \exp\left[-\frac{(t_i - t_{i-1})}{2D}\left(\frac{z_i - z_{i-1}}{t_i - t_{i-1}}\right)^2\right] \Delta z_i\right\}. \tag{5.6}$$

Consider now the probability $P[z(s)]$ that the particle followed a continuous path $z(s)$ from $z(t_0) = z_0$ to $z(t) = z$. This probability is obviously the continuum version of Eq. (5.6) with $z(t_i) = z_i$, $i = 0, ..., N - 1$ and $z(t_N) = z$:

$$P[z(s)] = \exp\left[-\frac{1}{2D}\int_{t_0}^{t} ds \left(\frac{dz}{ds}\right)^2\right] D_w[z(s)]. \tag{5.7}$$

The probability that the particle followed *some* path is equal to the integral of Eq. (5.7) over *all* paths and must be given by Eq. (5.5):

$$\int_{z}^{z+\Delta z} dz' \int_{paths} P[z(s)] = P(t, z)\Delta z, \tag{5.8}$$

where the sum is over all paths such that $z(0) = z_s$ and $z(t) = z'$. Equation (5.8) implies

$$\int D_w[z(s)]\exp\left[-\frac{1}{2D}\int_{t_0}^{t} ds\left(\frac{dz}{ds}\right)^2\right] = \left[\frac{1}{2\pi D(t-t_0)}\right]^{1/2}\exp\left[-\frac{(t-t_0)}{2D}\left(\frac{z-z_0}{t-t_0}\right)^2\right], \tag{5.9}$$

where the integration is over all paths $z(s)$ such that $z(t_0) = z_0$ and $z(t) = z$.

Consider now the calculation of the average value of a functional $F(z(s))$ of the path followed by the particle. This average is given by the functional integral

$$< F(z(s)) > = \frac{\int D_w[z(s)]\exp\left[-\frac{1}{2D}\int_{t_0}^{t} ds\left(\frac{dz}{ds}\right)^2\right] F(z(s))}{\int D_w[z(s)]\exp\left[-\frac{1}{2D}\int_{t_0}^{t} ds\left(\frac{dz}{ds}\right)^2\right]}. \tag{5.10}$$

If we take

$$F(z(s)) = \exp\frac{1}{D}\int_{t_0}^{t} ds\, V(s, z(s)), \tag{5.11}$$

we have

$$< F(z(s)) > = N\int D_w[z(s)]\exp\left\{-\frac{1}{D}\int_{t_0}^{t} ds\left[\frac{1}{2}\left(\frac{dz}{ds}\right)^2 - V(s, z(s))\right]\right\}, \tag{5.12}$$

where N is a normalization constant

$$N^{-1} = \int D_w[z(s)]\exp\left[-\frac{1}{2D}\int_{t_0}^{t} ds\left(\frac{dz}{ds}\right)^2\right]. \tag{5.13}$$

The functional integrals Eqs. (5.9), (5.10), and (5.12) are called Wiener integrals [72]. All the statistical aspects of Brownian motion can be expressed in terms of these integrals.

From the form of Eq. (5.12) we see the analogy between Wiener integrals and Feynman path integrals. A Feynman integral can be (formally) converted into a Wiener integral by analytic continuation in the mass, in the time, or in $\hbar$. Moreover, since they can be rigorously defined, Wiener integrals have often been the starting point of studies attempting to rigorously define Feynman path integrals.

The brevity of the preceding discussion really does not do justice to the application of Wiener integrals to the theory of Brownian motion. References 7, 8, and 17 contain richer, more satisfying discussions.

6. METHODS OF APPROXIMATION

In virtually every realistic situation the parabolic equation cannot be solved analytically and one must, therefore, resort to some method of approximation. The methods which are used in acoustics generally fall within two categories: asymptotic methods, such as ray acoustics, and perturbation methods, such as the Born approximation or the Rytov approximation. In this section we will use path integrals to develop two methods belonging to each category.

6.1 Straight-Line Geometric Optics

If acoustic energy is propagated from $(0, z_s)$ to (R, z), Eq. (3.27) becomes

$$\Psi(R, z|0, z_s) = \int D[z(r)] \exp\left\{ ik_o \int_0^R dr \left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 - U(r, z(r)) \right] \right\}, \tag{6.1a}$$

where $z(0) = z_s$ and $z(R) = z$. The integral is normalized so that

$$\int D[z(r)] \exp\left[\frac{ik_o}{2} \int_0^R dr \left(\frac{dz}{dr}\right)^2 \right] = \left(\frac{k_o}{2\pi iR}\right)^{1/2} \exp\left[\frac{ik_o R}{2} \left(\frac{z - z_s}{R}\right)^2 \right]. \tag{6.1b}$$

In straight-line geometric optics the assumption is made that a ray picture is valid and refraction is unimportant. In such a situation the acoustic phase depends only on the ray path which follows a straight line from source to receiver. The equation for this path is

$$z_{g.o.}(r) = z_s + \frac{r}{R}(z - z_s). \tag{6.2}$$

This suggests introducing a change of path variable

$$z(r) = z_{g.o.}(r) + z'(r) \tag{6.3}$$

and integrate over $z'(r)$ rather than $z(r)$. Since $z(0) = z_{g.o.}(0) = z_s$ and $z(R) = z_{g.o.}(R) = z$, the function z' satisfies the endpoint conditions

$$z'(0) = z'(R) = 0. \tag{6.4}$$

Moreover, the Jacobian of the transformation is unity. To see this, we return to the discrete version of the path integral. Equation (6.3) becomes

$$z_i = z_s + \frac{i}{N}(z - z_s) + z_i', \tag{6.5}$$

where $i = 0, \ldots, N$. Therefore

$$\int dz_1 \ldots dz_{N-1} = \int dz_1' \ldots dz_{N-1}', \tag{6.6}$$

since Eq. (6.5) corresponds to simply adding constants to the variables z_i. Equations (6.1) become

$$\Psi(R, z|0, z_s) = \exp\left[\frac{ik_oR}{2}\left(\frac{z - z_s}{R}\right)^2\right] \int D[z'(r)] \exp\left(\frac{ik_o}{2}\int_0^R dr\left\{\frac{1}{2}\left(\frac{dz'}{dr}\right)^2 - U\left[r, z_s + \frac{r}{R}(z - z_s) + z'(r)\right]\right\}\right) \tag{6.7a}$$

and

$$\int D[z'(r)] \exp\left[\frac{ik_o}{2}\int_0^R dr\left(\frac{dz'}{dr}\right)^2\right] = \left(\frac{k_o}{2\pi iR}\right)^{1/2} \exp\left\{\frac{ik_oR}{2}\left[\frac{z'(R) - z'(0)}{R}\right]^2\right\} = \left(\frac{k_o}{2\pi iR}\right)^{1/2}. \tag{6.7b}$$

So far we have made no approximations. We now assume that in the path integration of Eq. (6.7a) the dependence of U on z' can be ignored. That part of the integrand depending on U can then be moved outside the integral sign, and the integral over z' is just the normalization condition Eq. (6.7b). We obtain

$$\Psi(R, z|0, z_s) = \Psi_o(R, z|0, z_s) \exp\left[-ik_o\int_0^R dr\, U\left(r, z_s + \frac{r}{R}(z - z_s)\right)\right], \tag{6.8}$$

where Ψ_o is the solution to the parabolic equation with $n(r, z) = 1$ (i.e. with $U = 0$):

$$\Psi_o(R, z|0, z_s) = \left(\frac{k_o}{2\pi iR}\right)^{1/2} \exp\left[\frac{ik_oR}{2}\left(\frac{z - z_s}{R}\right)^2\right]. \tag{6.9}$$

The straight-line geometric-optics approximation is important for the study of electromagnetic propagation through the atmosphere but it is of little value for underwater sound propagation (at moderate to long ranges), because the sound channel always introduces significant refraction. However this example does illustrate an important methodological point. Many times it is not necessary to actually evaluate a path integral in order to obtain a desired result. In this example the normalization condition determined the only path integral needed.

6.2 Ray Acoustics

The path integral can be viewed as a coherent sum of elementary ray solutions $\exp[ik_oA(z(r))]$ corresponding to various paths $z(r)$. These ray paths do not satisfy any type of eikonal or ray equation however. They are completely arbitrary with the exception they must start at the source and end at the receiver. Consider what happens when k_o becomes large. Most of the paths will contribute little to the sum, because $\exp[ik_oA(z(r))]$ will oscillate rapidly and be canceled by the equally rapidly oscillating contributions from other nearby paths. The only exceptions will be for those paths $z(r)$ which, when perturbed ($z^* \to z^* + z'$), do not produce any significant change in k_oA. The difference between $A(z^*(r) + z'(r))$ and $A(z^*(r))$ will never be zero for all paths z', but it will be sufficiently small if it vanishes to first order in z' and if terms higher than second order are small. The situation is entirely analogous to that

encountered when one evaluates an integral by the method of stationary phase. The only difference is that one deals with an infinite-dimensional integral rather than a finite-dimensional integral.

Turning now to specifics, we introduce in Eqs. (6.1) the change of variable

$$z(r) = z^*(r) + z'(r) \tag{6.10a}$$

with

$$z'(0) = z'(R) = 0 \tag{6.10b}$$

and expand the exponent through second order in z':

$$\Psi(R, z|0, z_s) = \exp\,[ik_o A(z^*(r))] \int D[z'(r)] \exp\Bigg\{ik_o \times \int_0^R dr \left[\frac{dz^*}{dr}\frac{dz'}{dr} - z'\partial_z U(r, z^*) + \frac{1}{2}\left(\frac{dz'}{dr}\right)^2 - \frac{1}{2}(z')^2\partial_z^2 U(r, z^*)\right]\Bigg\}, \tag{6.11}$$

where

$$A(z^*(r)) = \int_0^R dr \left[\frac{1}{2}\left(\frac{dz^*}{dr}\right)^2 - U(r, z^*)\right]. \tag{6.12}$$

First-order cancellation requires

$$\int_0^R dr \left\{\frac{dz^*}{dr}\frac{dz'}{dr} - z'\partial_z U(r, z^*)\right\} = 0,$$

or, integrating by parts and using Eq. (6.10b),

$$\int_0^R dr\, z'(r)\left[\frac{d^2z^*}{dr^2} + \partial_z U(r, z^*)\right] = 0. \tag{6.13}$$

The only way Eq. (6.13) can be satisfied for all z' is for z^* to satisfy the equation

$$\frac{d^2z^*}{dr^2} = -\partial_z U(r, z^*). \tag{6.14}$$

Equation (6.14), together with the endpoint conditons $z^*(0) = z_s$ and $z^*(R) = z$ determine the ray path z^*. Equation (6.14) corresponds to Newton's second law $F = ma$. We now rewrite Eq. (6.11):

$$\Psi(R, z|0, z_s) = \exp\,[ik_o A(z^*(r)]\, T, \tag{6.15a}$$

where

$$T = \int D[z'(r)] \exp\left\{\frac{ik_o}{2}\int_0^R dr \left[\left(\frac{dz'}{dr}\right)^2 - (z')^2 M(r)\right]\right\}, \tag{6.15b}$$

with

$$M(r) \equiv \partial_z^2 U(r, z^*(r)). \tag{6.16}$$

Again, the path integral is normalized so that

$$\int D[z'(r)] \exp\left[\frac{ik_o}{2}\int_0^R dr\left(\frac{dz'}{dr}\right)^2\right] = \left(\frac{k_o}{2\pi iR}\right)^{1/2}. \tag{6.17}$$

Before proceeding with the evaluation of T, let us indicate how the straight-line geometric-optics approximation is recovered. We take $U(r, z)$ to be such a slowly varying function of z that $\partial_z U(r, z)$ in Eq. (6.14) and $M(r)$ in Eq. (6.15b) can be ignored. The solution to Eq. (6.14) is then $z_{g.o.}(r)$ of Eq. (6.2) and we immediately obtain Eq. (6.8).

The integral T can be written in the discrete form

$$T = \lim_{\substack{N\to\infty \\ \Delta r N = R}} T_N, \tag{6.18}$$

where (dropping the prime on the path)

$$T_N = \left(\frac{k_o}{2\pi i\Delta r}\right)^{N/2} \int dz_1 \ldots dz_{N-1}$$
$$\times \exp\left\{\frac{ik_o}{2\Delta r}\left[z_1^2 + (z_2 - z_1)^2 + \ldots + (z_{N-1} - z_{N-2})^2 + z_{N-1}^2\right]\right.$$
$$\left. - \frac{ik_o\Delta r}{2}\left(z_1^2 M_1 + z_2^2 M_2 + \ldots + z_{N-1}^2 M_{N-1}\right)\right\}, \tag{6.19}$$

with $M_i = M(i\Delta r) = \partial_z^2 U(i\Delta r, z^*(i\Delta r))$. Collecting terms in the exponent gives

$$T_N = \left(\frac{k_o}{2\pi i\Delta r}\right)^{N/2} \int dz_1 \ldots dz_{N-1} \exp\left(\frac{ik_o}{2\Delta r}\sum_{i,j=1}^{N-1} z_i A_{ij} z_j\right). \tag{6.20}$$

The matrix (A_{ij}) has components

$$A_{i,i} = 2 - (\Delta r)^2 M_i,$$
$$A_{i+1,i} = A_{i,i+1} = -1,$$

and

$$A_{i,j} = 0, \; |i - j| \geqslant 2. \tag{6.21}$$

We now introduce the subdeterminants

$$D_0 = 1$$

and

$$D_k = \det\,(A_{ij}), \quad i,j = 1, \ldots, k. \tag{6.22}$$

That is

$$D_1 = A_{11},$$
$$D_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix},$$
$$\ldots$$
$$D_{N-1} = \det A.$$

These determinants are called Gramians and are central to the Gram-Schmidt orthogonalization technique for constructing a set of linearly independent basis vectors. From Eq. (6.21) we have

$$D_k = A_{kk} D_{k-1} - A_{k,k-1}^2 D_{k-2} \tag{6.23}$$

for $k = 2, \ldots, N-1$.

For the present we will assume (A_{ij}) is a positive define matrix. It follows [73] there exists an orthogonal transformation of the variables z_i such that

$$\sum_{i,j=1}^{N-1} z_i A_{ij} z_j = \sum_{i=1}^{N-1} \left(\frac{D_i}{D_{i-1}} \right) w_i^2. \tag{6.24}$$

Since the transformation is orthogonal,

$$\int dz_1 \ldots dz_{N-1} = \int dw_1 \ldots dw_{N-1}. \tag{6.25}$$

Substituting Eqs. (6.24) and (6.25) into Eq. (6.20) gives

$$T_N = \left(\frac{k_o}{2\pi i \Delta r} \right)^{N/2} \int dw_1 \ldots dw_{N-1} \exp \left[\frac{ik_o}{2\Delta r} \sum_{i=1}^{N-1} \left(\frac{D_i}{D_{i-1}} \right) w_i^2 \right]. \tag{6.26}$$

This $(N-1)$ dimensional integral is the product of $N-1$ one-dimensional integrals, each of which may be evaluated using Eq. (4.21). We find

$$T_N = \left(\frac{k_o}{2\pi i \Delta r} \right)^{1/2} \prod_{i=1}^{N-1} \left(\frac{D_{i-1}}{D_i} \right)^{1/2} = \left(\frac{k_o}{2\pi i \Delta r (\det A)} \right)^{1/2}. \tag{6.27}$$

We now define

$$\begin{aligned} Q_0 &= 0, \\ Q_1 &= \Delta r D_0, \\ Q_2 &= \Delta r D_1, \\ &\ldots \\ Q_N &= \Delta r D_{N-1}, \end{aligned} \tag{6.28}$$

so that

$$T_N = \left(\frac{k_o}{2\pi i Q_N} \right)^{1/2}. \tag{6.29}$$

From Eqs. (6.21) and (6.23) we have the recursion relation

$$\left(\frac{Q_{k+1} - Q_k}{\Delta r} \right) - \left(\frac{Q_k - Q_{k-1}}{\Delta r} \right) = -\Delta r M_k Q_k \tag{6.30a}$$

for $k = 1, \ldots, N-1$. Moreover

$$Q_0 = 0 \tag{6.30b}$$

and

$$\frac{Q_1 - Q_0}{\Delta r} = 1. \tag{6.30c}$$

As $\Delta r \rightarrow 0$,

$$\{Q_0, \ldots, Q_N\} \rightarrow Q(r), \quad 0 \leqslant r \leqslant R,$$

and Eqs. (6.30) become, respectively,

$$\frac{d^2 Q(r)}{dr^2} + M(r) Q(r) = 0, \tag{6.31a}$$

$$Q(0) = 0, \tag{6.31b}$$

and

$$\frac{dQ}{dr}(0) = 1. \tag{6.31c}$$

We therefore obtain the result

$$\int D[z(r)] \exp\left\{\frac{ik_o}{2} \int_0^R dr \left[\left(\frac{dz}{dr}\right)^2 - z^2(r) M(r)\right]\right\} = \left[\frac{k_o}{2\pi i Q(R)}\right]^{1/2} \tag{6.32}$$

for $z(0) = z(R) = 0$, where $Q(r)$ satisfies Eqs. (6.31). Equation (6.32) is valid provided $Q(R) \neq 0$, i.e., provided (A_{ij}) is a positive definite matrix. This example illustrates the close connection between Feynman path integrals and boundary-value problems of the Sturm-Liouville type [57].

We must now solve Eqs. (6.31) for $M(r) = \partial_z^2 U(r, z^*(r))$. The ray equation, Eq. (6.14), may be replaced by the equations

$$z^*(r) = z_s + \int_0^r ds\, p(s) \tag{6.33a}$$

and

$$p(r) = p(0) - \int_0^r ds\, \partial_z U(s, z^*(s)), \tag{6.33b}$$

where

$$p(r) = \frac{dz^*(r)}{dr}. \tag{6.33c}$$

Physically $p(r) = \tan\theta(r), \simeq \theta(r)$ where $\theta(r)$ is the local angle the ray path $z^*(r)$ makes with respect to the horizontal. Consider the quantity $dz^*(r)/dp(0)$. Differentiating Eq. (6.33a) gives

$$\frac{dz^*(r)}{dp(0)} = \int_0^r ds \frac{dp(s)}{dp(0)}. \tag{6.34a}$$

But from Eq. (6.33b) we have

$$\frac{dp(s)}{dp(0)} = 1 - \int_0^s ds'\, \partial_z^2 U(s', z^*(s')) \frac{dz^*(s')}{dp(0)}. \tag{6.34b}$$

Substituting Eq. (6.34b) into (6.34a) gives the integral equation

$$\frac{dz^*(r)}{dp(0)} = r - \int_0^r ds \int_0^s ds'\, \partial_z^2 U(s', z^*(s')) \frac{dz^*(s')}{dp(0)}. \tag{6.34c}$$

If we differentiate Eq. (6.34c) with respect to r, we obtain

$$\frac{d}{dr}\left[\frac{dz^*(r)}{dp(0)}\right] = 1 - \int_0^r ds\, \partial_z^2 U(s, z^*(s))\, \frac{dz^*(s)}{dp(0)} \tag{6.34d}$$

and

$$\frac{d^2}{dr^2}\left[\frac{dz^*(r)}{dp(0)}\right] = -\partial_z^2 U(r, z^*(r))\, \frac{dz^*(r)}{dp(0)} = -M(r)\, \frac{dz^*(r)}{dp(0)}. \tag{6.34e}$$

From Eqs. (6.34c) and (6.34d) we have

$$\left.\frac{dz^*(r)}{dp(0)}\right|_{r=0} = 0 \tag{6.34f}$$

and

$$\left.\frac{d}{dr}\left[\frac{dz^*(r)}{dp(0)}\right]\right|_{r=0} = 1. \tag{6.34g}$$

Comparing Eqs. (6.34e), (6.34f), and (6.34g) with Eqs. (6.31), we see

$$Q(r) = \frac{dz^*(r)}{dp(0)}. \tag{6.35}$$

Combining Eqs. (6.15), (6.32), and (6.35), we finally find

$$\Psi(R, z|o, z_s) = \left[\frac{k_o}{2\pi i dz/dp(0)}\right]^{1/2} \exp\, [ik_o A(z^*(r))], \tag{6.36}$$

where $p(0)$ is defined by Eq. (6.33c), A by Eq. (6.12), and the ray path z^* by Eq. (6.14).

If the matrix (A_{ij}) of Eqs. (6.20) and (6.21) is not positive definite, det $A = Q(R) = dz/dp(0) = 0$. The endpoint $z^*(R) = z$ of the ray path is insensitive to the initial grazing angle, which means the ray path is degenerate, and a caustic is present. This leads to an infinite amplitude, and the ray acoustics approximation breaks down.

We can go further. Returning to Eqs. (6.27) and (6.28), we have

$$\frac{1}{dz/dp(0)} = \lim_{N\to\infty}\left[\frac{1}{Q_1}\,\frac{Q_1}{Q_2}\,\frac{Q_2}{Q_3}\cdots\frac{Q_{N-1}}{Q_N}\right]$$

$$= \lim_{N\to\infty}\left\{\left[\frac{1}{dz^*(r_1)/dp(0)}\right]\left[\frac{dz^*(r_1)/dp(0)}{dz^*(r_2)/dp(0)}\right]\cdots\left[\frac{dz^*(r_{N-1})/dp(0)}{dz^*(r_N)/dp(0)}\right]\right\} \tag{6.37}$$

$$= (-1)^l \lim_{N\to\infty}\left[\left|\frac{1}{dz^*(r_1)/dp(0)}\right|\cdots\left|\frac{dz^*(r_{N-1})/dp(0)}{dz^*(r_N)/dp(0)}\right|\right], \tag{6.38}$$

where l is the number of the factors

$$\frac{dz^*(r_{i-1})/dp(0)}{dz^*(r_i)/dp(0)}$$

which are negative. If this ratio is negative, $dz^*(r_{i-1})/dp(0)$ and $dz^*(r_i)/dp(0)$ are opposite in sign, implying that somewhere in the range r_{i-1} to r_i, $dz^*/dp(0) = 0$, (a caustic was encountered). Therefore, as $N \to \infty$,

$$\left[\frac{1}{dz/dp(0)}\right]^{1/2} = \exp\left[\frac{-i\pi l}{2}\right]\left|\frac{1}{dz/dp(0)}\right|^{1/2}, \tag{6.39}$$

where l is the number of caustics encountered as one propagates from 0 to R. Consider now Eq. (6.12) for $A(z^*(r))$:

$$A(z^*(r)) = \lim_{\Delta r \to 0} \Delta r \sum_{i=1}^{N} \left[\frac{1}{2}\left(\frac{z_i^* - z_{i-1}^*}{\Delta r}\right)^2 - U(r_i, z_i^*)\right].$$

Remembering that $z_0^* = z_s$, we have

$$\frac{\partial A(z^*(r))}{\partial z_s} = -\lim_{\Delta r \to 0}\left[\frac{z_1^* - z_0^*}{\Delta r}\right] = -p(0), \tag{6.40}$$

so that

$$\frac{1}{dz/dp(0)} = \frac{dp(0)}{dz} = -\frac{\partial^2 A(z^*(r))}{\partial z \partial z_s}. \tag{6.41}$$

Combining Eqs. (6.36), (6.39), and (6.41) gives

$$\Psi(R, z|0, z_s) = \left[\frac{k_o}{2\pi i}\left|\frac{\partial^2 A(z^*)}{\partial z \partial z_s}\right|\right]^{1/2} \exp\left[ik_o A(z^*) - \frac{i\pi l}{2}\right], \tag{6.42}$$

where again $\exp(-i\pi l/2)$ represents the cumulative effect of phase jump of $\pi/2$ on passing through each caustic.

We have assumed until now that a single solution z^* exists to the ray equation. If several distinct rays contribute, Ψ is given by a sum of terms of the form of (6.42):

$$\Psi(R, z|0, z_s) = \sum_j \left[\frac{k_o}{2\pi i}\left|\frac{\partial^2 A(z_j^*)}{\partial z \partial z_s}\right|\right]^{1/2} \exp\left[ik_o A(z_j^*) - \frac{i\pi l_j}{2}\right]. \tag{6.43}$$

(The integer l_j depends on the particular ray path.) For two solutions z_i^* and z_j^* to the ray equation to be considered distinct rays, the phase difference $k_o|A(z_i^*) - A(z_j^*)|$ must be greater than some fraction of a cycle, say $\pi/2$:

$$|A(z_i^*) - A(z_j^*)| > (\text{acoustic wavelength})/4. \tag{6.44}$$

We have spent considerable space on ray acoustics because it is probably the most important approximation and it gave us the opportunity to evaluate an important path integral (Eq. (6.32)).

6.3 Standard Perturbation Theory

We start our development of perturbation methods by expanding that part of the exponential in Eq. (6.1a) which depends on U:

$$\exp\left[-ik_o \int_0^R dr\, U(r, z(r))\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-ik_o)^n}{n!} \int_0^R dr_1 \ldots \int_0^R dr_n\, U(r_1, z(r_1)) \ldots U(r_n, z(r_n)). \tag{6.45}$$

By introducing the Fourier transform

$$U(r, z) = \int \frac{dk}{(2\pi)} e^{ikz} U(r, k), \tag{6.46}$$

the right-hand side of Eq. (6.45) becomes

$$\sum_{n=0}^{\infty} \frac{(-ik_o)^n}{n!} \int_0^R dr_1 \ldots dr_n \int \frac{dk_1}{(2\pi)} \cdots \frac{dk_n}{(2\pi)} U(r_1, k_1) \ldots U(r_n, k_n)$$

$$\times \exp ik_o \int_0^R dr\, z(r) M(r), \tag{6.47}$$

where $M(r)$ is a sum of δ functions:

$$M(r) = \frac{1}{k_o} \sum_{i=1}^{n} k_i \delta(r - r_i). \tag{6.48}$$

Substituting this expansion into Eq. (6.1a) gives

$$\Psi(R, z|0, z_s) = \sum_{n=0}^{\infty} \frac{(-ik_o)^n}{n!} \int_0^R dr_1 \ldots dr_n \int \frac{dk_1}{(2\pi)} \cdots \frac{dk_n}{(2\pi)} U(r_1, k_1) \ldots U(r_n, k_n)$$

$$\times \int D[z(r)] \exp\left\{ik_o \int_0^R dr\left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 + z(r)M(r)\right]\right\}. \tag{6.49}$$

($M(r)$ depends on $r_1, \ldots, r_n, k_1, \ldots, k_n$.) Although we will later evaluate this path integral directly, it is convenient at this stage to observe Eq. (6.49) has the structure

$$\Psi = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^R dr_1 \ldots \int_0^R dr_n f(r_1, \ldots, r_n), \tag{6.50a}$$

where f is symmetric under permutations of the r_i's. Since there are $n!$ of these permutations, Eq. (6.50a) may be written as

$$\Psi = \sum_{n=0}^{\infty} \int_0^R dr_n \int_0^{r_n} dr_{n-1} \ldots \int_0^{r_2} dr_1\, f(r_1, \ldots r_n). \tag{6.50b}$$

The result of rewriting the expression for Ψ in the form of Eq. (6.50b) is that the variables $r_1, \ldots, r_n$ which appear in $M(r)$ become *ordered*:

$$0 \leq r_1 \leq r_2 \leq \ldots \leq r_{n-1} \leq r_n \leq R. \tag{6.51}$$

With this ordering the path integral in Eq. (6.49) can be expressed in a particularly convenient form. Writing $T_M(R, z|0, z_s)$ for this path integral, we first note it is the solution to the parabolic equation with $U(r, z) = -zM(r)$ and hence obeys the composition law. Suppose we iterate the composition law and pick as intermediate range points the variables $r_1, \ldots, r_n$:

$$T_M(R, z|0, z_s) = \int dz_1 \ldots dz_n\, T_M(R, z|r_n, z_n)$$

$$\times T_M(r_n, z_n|r_{n-1}, z_{n-1}) \ldots T_M(r_1, z_1|0, z_s). \tag{6.52}$$

(Remember the composition law by itself is exact.) We now write a path-integral solution for the factors in Eq. (6.52):

$$T_M(r_1, z_1|0, z_s) = \int D[z(r)] \exp\left\{ik_o \int_0^{r_1} dr \left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 + z(r)M(r)\right]\right\}, \tag{6.53a}$$

where $z(0) = z_s$, and $z(r_1) = z_1$;

$$T_M(r_n, z_n|r_{n-1}, z_{n-1}) = \int D[z(r)] \exp\left\{ik_o \int_{r_{n-1}}^{r_n} dr \left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 + zM\right]\right\}, \tag{6.53b}$$

with $z(r_{n-1}) = z_{n-1}$, and $z(r_n) = z_n$; and

$$T_M(R, z|r_n, z_n) = \int D[z(r)] \exp\left\{ik_o \int_{r_n}^{R} dr \left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 + zM\right]\right\}, \tag{6.53c}$$

with $z(r_n) = z_n$, and $z(R) = z$. These expressions differ only in the endpoint conditions satisfied by the paths. We now focus on the integration over M which appears in Eqs. (6.53). For Eq. (6.53a) we have

$$\int_0^{r_1} dr\, z(r)M(r) = \frac{1}{k_o} \sum_{i=1}^{n} k_i z(r_i) \int_0^{r_1} dr\, \delta(r - r_i) = \frac{k_1 z(r_1)}{2k_o} \tag{6.54}$$

since the r_i's are ordered and

$$\int_0^{r_1} dr\, \delta(r - r_1) = \frac{1}{2}.$$

Therefore

$$\begin{aligned} T_M(r_1, z_1|0, z_s) &= e^{ik_1 z(r_1)/2} \int D[z(r)] \exp\left[\frac{ik_o}{2} \int_0^{r_1} dr \left(\frac{dz}{dr}\right)^2\right] \\ &= e^{ik_1 z_1/2} \left(\frac{k_o}{2\pi i r_1}\right)^{1/2} \exp\left[\frac{ik_o r_1}{2} \left(\frac{z_1 - z_s}{r_1}\right)^2\right] \\ &= e^{ik_1 z_1/2}\, \Psi_o(r_1, z_1|0, z_s). \end{aligned} \tag{6.55a}$$

Again Ψ_o is the Green's function with $U = 0$. Further,

$$T_M(r_n, z_n|r_{n-1}, z_{n-1}) = e^{i(k_n z_n + k_{n-1} z_{n-1})/2}\, \Psi_o(r_n, z_n|r_{n-1}, z_{n-1}) \tag{6.55b}$$

and

$$T_M(R, z|r_n, z_n) = e^{ik_n z_n/2} \Psi_o(R, z|r_n, z_n), \tag{6.55c}$$

giving

$$\begin{aligned} T_M(R, z|0, z_s) = \int dz_1 \ldots dz_n \exp\left(i \sum_{i=1}^{N} k_i z_i\right) \Psi_o(R, z|r_n, z_n) \\ \times \Psi_o(r_n, z_n|r_{n-1}, z_{n-1}) \ldots \Psi_o(r_1, z_1|0, z_s). \end{aligned} \tag{6.56}$$

Combining Eqs. (6.49), (6.52), and (6.56) and using Eq. (6.51), we have

$$\Psi(R, z|0, z_s) = \sum_{n=0}^{\infty} (-ik_o)^n \int_0^R dr_n \int_0^{r_n} r_{n-1} \ldots \int_0^{r_2} dr_1 \int dz_1 \ldots \int dz_n$$
$$\times\ U(r_n, z_n) U(r_{n-1}, z_{n-1}) \ldots U(r_1, z_1)$$
$$\times\ \Psi_o(R, z|r_n, z_n)\Psi_o(r_n, z_n|r_{n-1}, z_{n-1}) \ldots \Psi_o(r_1, z_1|0, z_s), \quad (6.57)$$

where we have used Eq. (6.46) to do the integrals over the k_i's. This expression is the standard Neumann series which expresses Ψ as a power series in U. If Ψ is approximated by keeping only the first $n + 1$ terms, one is using nth-order perturbation theory. To zeroth order

$$\Psi(R, z|0, z_s) = \Psi_o(R, z|0, z_s). \quad (6.58)$$

If two terms are kept, one is making the Born approximation

$$\Psi(R, z|0, z_s) = \Psi_o(R, z|0, z_s) - ik_o \int_0^R dr_1 \int dz_1$$
$$\times\ \Psi_o(R, z|r_1, z_1) U(r_1, z_1) \Psi_o(r_1, z_1|0, z_s). \quad (6.59)$$

The second term in Eq. (6.59) is called the Born term.

In the preceeding we have expressed Ψ as a power series in U. It is also possible to do perturbation theory around a part of U. Suppose

$$U(r, z) = U_o(r, z) + U'(r, z), \quad (6.60)$$

where U' is in some sense small and the parabolic equation could be solved if it were absent. We might then expand that part of the exponential in Eq. (6.1a) which depends on U' and proceed as before. We would obtain

$$\Psi(R, z|0, z_s) = \Psi_{U_o}(R, z|0, z_s) - ik_o \int_0^R dr_1 \int dz_1$$
$$\times\ \Psi_{U_o}(R, z|r_1, z_1) U'(r_1, z_1) \Psi_{U_o}(r_1, z_1|0, z_s) + \ldots, \quad (6.61)$$

where Ψ_{U_o} satisfies the equations

$$-2ik_o\partial_r \Psi_{U_o}(r, z|r', z') = \left[\partial_z^2 - 2k_o^2 U_o(r, z)\right] \Psi_{U_o}(r, z|r', z') \quad (6.62a)$$

and

$$\Psi_{U_o}(r, z|r, z') = \delta(z - z'). \quad (6.62b)$$

If Eqs. (6.62) cannot be solved exactly, Eq. (6.61) still might be useful if Ψ_{U_o} is approximated using some other technique, say ray acoustics.

Before going on to the Rytov approximation, we mention one minor point which may have given trouble to some readers. In writing Eq. (6.49), we moved the integrals over the r_i's outside the integral over the path $z(r)$. By use of the discrete representation for the path integral, it is obvious this interchange is permitted.

6.4 The Rytov Approximation

In the previous subsection we developed an expansion for Ψ in powers of U. In certain situations it is preferable to use a perturbation scheme based on an expansion of $\log_e\Psi$ in

powers of U. This scheme is called by Fradkin [12] "modified perturbation theory." If in this expansion one keeps only the first nontrivial term, one has made the Rytov approximation [74-76]. The Rytov approximation essentially amounts to exponentiating the born term. In this subsection we show how it may be obtained using the path-integral formalism.

We wish to evaluate the path integral in Eq. (6.49), which we have called T_M, without ordering the r_i variables. We now have three techniques for doing this. We can use the velocity representation and perform a translation in $v(r)$, we can use the discrete representation and carry out an analysis smilar to that which led to the evaluation of the path integral in Eq. (6.15b), or we can use ray acoustics directly. Ray acoustics amounted to expanding the action (eikonal) $A(z(r))$ through second order about the ray trajectory z^*. If A is only quadratic in $z(r)$, which will be the case if $U(r, z)$ is only quadratic in z, independent of its dependence or r, then ray acoustics gives the exact solution. The integral T_M is the solution for Ψ with $U(r, z) = -zM(r)$. Therefore ray acoustics directly gives the value for T_M.

Although it is instructive to evaluate T_M using all three techniques, we shall only record its evaluation using the velocity representation. We have

$$T_M = \int D[v(r)]\,\delta\left[z - z_s - \int_0^R dr\, v(r)\right] \times \exp\left\{ik_o \int_0^R dr \left[\frac{1}{2} v^2 + M(r) z_s + M(r) \int_0^r ds\, v(s)\right]\right\}. \tag{6.63}$$

We next let

$$v(r) = \frac{z - z_s}{R} + v'(r), \tag{6.64}$$

giving

$$T_M = \exp\left\{\frac{ik_o R}{2}\left(\frac{z - z_s}{R}\right)^2 + ik_o \int_0^R dr\, M(r)\left[z_s + \frac{r}{R}(z - z_s)\right]\right\} T_M', \tag{6.65a}$$

where

$$T_M' = \int D[v'(r)]\,\delta\left[\int_0^R dr\, v'(r)\right] \exp\left\{ik_o \int_0^R dr\left[\frac{1}{2} v'^2 + M(r)\int_0^r ds\, v'(s)\right]\right\}. \tag{6.65b}$$

We now use the relation

$$\int_0^R dr\, M(r) \int_0^r ds\, v'(s) = -\int_0^R dr\, v'(r) \tilde{M}(r), \tag{6.66}$$

where

$$\tilde{M}(r) = \int_0^r ds\, M(s). \tag{6.67}$$

Equation (6.66) follows from the constraint

$$\int_0^R dr\, v'(r) = 0.$$

Substituting Eq. (6.66) into T_M' and dropping the prime on $v'(r)$ gives

$$T_M' = \int D[v(r)]\, \delta\left[\int_0^R dr v(r)\right] \exp\left\{ik_o \int_0^R dr\left[\frac{1}{2} v^2(r) - \tilde{M}(r)v(r)\right]\right\}. \tag{6.68}$$

Equation (6.68) is easily evaluated using the transformation

$$v(r) = w(r) + \tilde{M}(r). \tag{6.69}$$

We find

$$T_M' = \left[\frac{k_o}{2\pi iR}\right]^{1/2} \exp\left[\frac{ik_oR}{2}\left\{\left[\frac{1}{R}\int_0^R dr\, \tilde{M}(r)\right]^2 - \frac{1}{R}\int_0^R dr\, \tilde{M}^2(r)\right\}\right]. \tag{6.70}$$

Collecting expressions yields

$$\int D[z(r)] \exp\left\{ik_o \int_0^R dr\left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 + z(r)M(r)\right]\right\}$$

$$= \Psi_o(R, z|0, z_s) \exp\left\{ik_o \int_0^R dr\, M(r)\left[z_s + \frac{r}{R}(z - z_s)\right]\right\}$$

$$\times \exp\left[\frac{ik_oR}{2}\left\{\left[\frac{1}{R}\int_0^R dr\, \tilde{M}(r)\right]^2 - \frac{1}{R}\int_0^R dr\, \tilde{M}^2(r)\right\}\right], \tag{6.71}$$

where $\tilde{M}$ is defined by Eq. (6.67). We have thus evaluated another important path integral.

With $M(r)$ defined by Eq. (6.48) we find

$$\int_0^R dr\, M(r)\left[z_s + \frac{r}{R}(z - z_s)\right] = \frac{1}{k_o}\sum_{i=1}^n k_i\left[z_s + \frac{r_i}{R}(z - z_s)\right], \tag{6.72a}$$

$$\frac{1}{R}\int_0^R dr\, \tilde{M}(r) = \int_0^R dr\, M(r)\left[1 - \frac{r}{R}\right] = \frac{1}{k_o}\sum_{i=1}^n k_i\left[1 - \frac{r_i}{R}\right], \tag{6.72b}$$

and

$$\frac{1}{R}\int_0^R dr\, (\tilde{M}(r))^2 = \int_0^R ds\, M(s)\int_0^R ds' M(s')\,\frac{1}{R}\int_0^R dr\, \theta(r - s)\theta(r - s')$$

$$= \frac{1}{k_o^2}\sum_{i,j=1}^n k_ik_j\left[1 - \frac{\max(r_i; r_j)}{R}\right], \tag{6.72c}$$

so that

$$\left[\frac{1}{R}\int_0^R dr\, \tilde{M}(r)\right]^2 - \frac{1}{R}\int_0^R dr\, \tilde{M}^2(r)$$

$$= -\frac{1}{(k_oR)^2}\sum_i k_i^2 r_i(R - r_i)$$

$$+ \frac{2}{(k_oR)^2}\sum_{i>j} k_ik_j[r_ir_j + R\max(r_i, r_j) - R(r_i + r_j)]. \tag{6.72d}$$

Substituting into T_M (Eq. 6.71) gives

$$T_M = \Psi_o(R, z|0, z_s) \exp\left\{ i \sum_i \left\{ k_i \left[z_s + \frac{r_i}{R}(z - z_s) \right] - \frac{1}{2}\frac{k_i^2}{k_o} r_i \left(1 - \frac{r_i}{R}\right) \right\} \right\}$$

$$\times \exp\left\{ \frac{i}{k_o R} \sum_{i>j} k_i k_j \left[r_i r_j - R \min(r_i, r_j) \right] \right\}. \tag{6.73}$$

One can check the validity of Eq. (6.73) by noting that if the r_i's are ordered, so that min $(r_i, r_j) = r_j (i > j)$, Eq. (6.73) must agree with Eq. (6.56). This check is a straightforward exercise which is easily carried out by Fourier transforming with respect to the depth variables all the Ψ's which appear in Eq. (6.56). The integrals over the z_i's then yield δ functions which permit all but one of the integrations over the Fourier transform variables to be carried out. The last integration is evaluated using Eq. (4.21). The final result agrees with Eq. (6.73).

The Rytov approximation follows from ignoring in T_M the cross terms involving $k_i k_j (i \neq j)$. Dropping these terms and substituting Eq. (6.73) back into (6.49), we obtain

$$\Psi(R, z|0, z_s) = \Psi_o(R, z|0, z_s) \sum_{n=0}^{\infty} \frac{(-ik_o)^n}{n!} \int_0^R dr_1 \ldots dr_n \int \frac{dk_1}{(2\pi)} \cdots \frac{dk_n}{(2\pi)}$$

$$\times U(r_1, k_1) \ldots U(r_n, k_n)$$

$$\times \exp\left\{ i \sum_{i=1}^{n} \left\{ k_i \left[z_s + \frac{r_i}{R}(z - z_s) \right] - \frac{1}{2}\frac{k_i^2}{k_o} r_i \left(1 - \frac{r_i}{R}\right) \right\} \right\}$$

$$= \Psi_o(R, z|0, z_s) \sum_{n=0}^{\infty} \frac{1}{n!} \left[-ik_o \int_0^R dr \int \frac{dk}{(2\pi)} U(r, k) \right.$$

$$\left. \times \exp\left\{ ik \left[z_s + \frac{r}{R}(z - z_s) \right] - \frac{ik^2}{2k_o} r \left(1 - \frac{r}{R}\right) \right\} \right]^n. \tag{6.74}$$

The $2n$-dimensional integrals factor into n identical two-dimensional integrals, which permits the infinite series to be summed giving the simple expression

$$\Psi(R, z|0, z_s) = \Psi_o(R, z|0, z_s) \exp X, \tag{6.75a}$$

where

$$X = -ik_o \int_0^R dr \int \frac{dk}{(2\pi)} U(r, k)$$

$$\times \exp\left\{ ik \left[z_s + \frac{r}{R}(z - z_s) \right] - \frac{ik^2}{2k_o} r \left(1 - \frac{r}{R}\right) \right\}. \tag{6.75b}$$

Before proceeding, we note that if we drop the k^2 term in Eq. (6.75b), we recover straight-line geometric optics:

$$X = -ik_o \int_0^R dr\, U\left\{ r, z_s + \frac{r}{R}(z - z_s) \right\}. \tag{6.76}$$

Equation (6.75b) for X may be simplified by using

$$U(r,k) = \int dz_1 e^{-iz_1 k} U(r, z_1) \tag{6.77}$$

and performing the integration over k with the help of Eq. (4.21):

$$X = -ik_o \int_0^R dr_1 \int dz_1\, U(r_1, z_1) \left[\frac{k_o R}{2\pi i r_1 (R - r_1)}\right]^{1/2}$$
$$\times \exp\left\{\frac{ik_o}{2r_1 R(R - r_1)} [R(z_s - z_1) + r_1 (z - z_s)]^2\right\}. \tag{6.78}$$

Since

$$[R(z_s - z_1) + r_1(z - z_s)]^2 = R(R - r_1)(z_1 - z_s)^2$$
$$+ r_1 R(z - z_1)^2 - r_1(R - r_1)(z - z_s)^2, \tag{6.79}$$

we have

$$X = \frac{-ik_o}{\Psi_o(R, z|0, z_s)} \int_0^R dr_1 \int dz_1\, \Psi_o(R, z|r_1, z_1) U(r_1, z_1) \Psi_o(r_1, z_1|0, z_s), \tag{6.80}$$

giving the Rytov approximation for the field Ψ:

$$\Psi(R, z|0, z_s) = \Psi_o(R, z|0, z_s) \exp\left[\frac{-ik_o}{\Psi_o(R, z|0, z_s)} \int_0^R dr_1 \int dz_1\right.$$
$$\left.\times \Psi_o(R, z|r_1, z_1)\, U(r_1, z_1) \Psi_o(r_1, z_1|0, z_s)\right]. \tag{6.81}$$

Comparing Eq. (6.81) with the Born approximation, Eq. (6.59), we see that in the Rytov approximation the field has the form

$$\Psi = \Psi_o \exp\left[\frac{\text{Born term}}{\Psi_o}\right]. \tag{6.82}$$

The Rytov approximation may be generalized by breaking U up according to Eq. (6.60) and treating U' as a perturbation. The final result is

$$\Psi = \Psi_{U_o} \exp\left[\frac{\text{modified Born term}}{\Psi_{U_o}}\right], \tag{6.83}$$

where Ψ_{U_o} is given by Eqs. (6.62) and the modified Born term is the second term on the right-hand side of Eq. (6.61).

Equation (6.83) is an important result. If U_o is the contribution due to the mean sound speed and U' represents the effect of random ocean variability on the sound speed, then Eq. (6.83) gives an adequate description of the propagation process over most of the region in range and frequency where one is interested in doing passive listening. Moreover Eq. (6.83) is amenable to the direct calculation of various coherence functions. However, one would probably want to include cross-range dependence by using the three-dimensional parabolic equation, and the numerical integrations which would be necessary are not trivial.

The work in this section points out what seems to be a general characteristic of Feynman path integrals. High-frequency approximations, where some type of ray picture emerges, are

natural to derive using path integrals. It is rather awkward, on the other hand, to use the formalism to develop perturbation schemes.

7. BOUNDARY CONDITIONS

So far we have assumed that the medium is infinite. Now we come to the crucial problem of incorporating into the analysis realistic boundary conditions. Although our general understanding of this problem is rather unsatisfactory at present, the path-integral formalism does give some indication of the direction future research might take.

The goal here is to use the formalism to develop algorithms for numerically solving the parabolic equation. With the inclusion of boundaries, many of the previously derived expressions do not apply, and it is almost hopeless to obtain even an approximate solution without detailed computer calculations. We are always assuming the medium possesses acoustic characteristics which are range dependent. If these characteristics are not present or can be ignored, then there is no compelling reason for introducing the parabolic equation in the first place.

The boundaries to be considered are the surface of the ocean and its bottom. The surface of the ocean does not create any particular problems, because it is almost always assumed in long-range-propagation modeling to be a free, plane surface which does not vary with time. When the ocean bottom is considered, a distinction must be made between propagation through water with a depth excess and bottom-limited or shallow-water propagation. In the case of a depth excess, the exact boundary conditions imposed on the pressure at the bottom interface are not so important over long propagation distances because acoustic energy which interacts with the bottom is scattered or absorbed and can for the most part be ignored. (This statement requires some modification if the receiving hydrophone or array is bottom mounted.) The bottom can therefore be modeled phenomenologically as being perfectly absorbing, and discontinuities in density and sound speed can be ignored. For such a situation it is relatively easy to solve the parabolic equation using the split-step Fourier algorithm. The only range-dependent quantity of interest is the sound speed.

For either shallow-water propagation or propagation through bottom-limited water, the situation is far more complicated. The bottom has a pronounced effect on the received signal and cannot be modeled as casually as in the case of a depth excess. One must deal realistically with the acoustic parameters of the bottom material, including absorption and discontinuities in density and sound speed. Moreover, variations in range of the water depth and of the bottom material become important, as well as the range variations of the speed of sound in the water. There does not seem to be an algorithm for solving the parabolic equation in this situation. All we can do in this review is indicate how path-integral techniques may be used to develop one.

7.1. Water with a Depth Excess: The Split-Step Fourier Algorithm

If the acoustic energy is constrained to a channel of depth L, then the composition law takes the form

$$\Psi(r, z|r', z') = \int_0^L dz_s \, \Psi(r, z|s, z_s)\Psi(s, z_s|r', z'). \tag{7.1}$$

It is easy to argue physically for the limits of integration in Eq. (7.1). The composition law is simply the mathematical statement that the propagation process can be viewed as occurring in two steps. Sound propagates first from the source at (r', z') to an intermediate point (s, z_s). The function $\Psi(s, z_s|r', z')$ is the source strength of a secondary sound field which originates at the intermediate point and propagates on to the receiver at (r, z). The field at (r, z) is the coherent sum of the contributions from all possible intermediate points at the particular range coordinate s. Consequently, to determine the limits of the integration in the composition law, one need only determine those points which could serve as the coordinates of an acoustic source (those depth positions where one could conceivably place a source). For the channel being considered, a source could be placed anywhere in the range $0 < z_s < L$.

If we take the intermediate point close to the receiver, we have, with an obvious change of notation,

$$\Psi(r + \Delta r, z|\epsilon, z_0) = \int_0^L dz' \, \Psi(r + \Delta r, z|r, z')\Psi(r, z'|\epsilon, z_0). \tag{7.2}$$

Any marching algorithm which determines the field at $r + \Delta r$ from the field at r will necessarily involve finding some expression for $\Psi(r + \Delta r, z|r, z')$ and then evaluating the integral over z' in Eq. (7.2).

We previously concluded that for Δr sufficiently small $\Psi(r + \Delta r, z|r, z')$ is approximately equal to the solution to the parabolic equation with constant index of refraction. That is,

$$\Psi(r + \Delta r, z|r,z') = \Psi_o(r + \Delta r, z|r, z') \exp\left\{\frac{i\Delta r k_o}{2} [n^2(r, z) - 1]\right\}, \tag{7.3}$$

where Ψ_o is the solution to the parabolic equation with $n(r, z) = 1$:

$$-2ik_o\partial_r\Psi_o(r, z|r', z') = \partial_z^2\Psi_o(r, z|r', z') \tag{7.4a}$$

and

$$\Psi_o(r', z|r', z') = \delta(z - z'). \tag{7.4b}$$

Both fields, Ψ and Ψ_o, satisfy the same boundary conditions at $z = 0$ and $z = L$. These conditions are in turn the same as those satisfied by the acoustic pressure. To derive the split-step Fourier algorithm, one assumes that L is the depth of the water column plus an artificially introduced bottom layer having a depth 1/4 to 1/3 the depth of the water and that the field vanishes at $z = 0$ and at $z = L$:

$$\Psi_o(r, 0|r', z') = 0 \tag{7.5a}$$

and

$$\Psi_o(r, L|r', z') = 0. \tag{7.5b}$$

Equations (7.4) and (7.5) are easy to solve. One expands the field Ψ_o in terms of a complete set of eigenfunctions which satisfy Eqs. (7.5):

$$\Psi_o(r, z|r', z') = \sum_{k=1}^{\infty} a_k(r, r', z') \left[\left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{k\pi z}{L}\right)\right]. \tag{7.6}$$

Substituting Eq. (7.6) into Eq. (7.4a) and using the orthonormality property

$$\left(\frac{2}{L}\right) \int_0^L dz \sin\left(\frac{k\pi z}{L}\right) \sin\left(\frac{k'\pi z}{L}\right) = \delta_{kk'}, \tag{7.7}$$

we get

$$2ik_o\partial_r a_k(r, r', z') = \left(\frac{k\pi}{L}\right)^2 a_k(r, r', z'), \tag{7.8a}$$

or

$$a_k(r, r', z') = a_k(r', r', z') \exp\left[-\frac{i(r - r')}{2k_o}\left(\frac{k\pi}{L}\right)^2\right]. \tag{7.8b}$$

Since we have the completeness property

$$\left(\frac{2}{L}\right)\sum_{k=1}^{\infty} \sin\left(\frac{k\pi z}{L}\right) \sin\left(\frac{k\pi z'}{L}\right) = \delta(z - z'), \tag{7.9}$$

Eq. (7.4b) will be satisfied if

$$a_k(r', r', z') = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{\pi z' k}{L}\right), \tag{7.10}$$

independent of r'. Combining Eqs. (7.3), (7.6), (7.8b), and (7.10), we find

$$\begin{aligned}\Psi(r + \Delta r, z | r, z') = \exp&\left\{\frac{i\Delta r k_o}{2}[n^2(r, z) - 1]\right\}\\ &\times \left(\frac{2}{L}\right)\sum_{k=1}^{\infty} \sin\left(\frac{k\pi z}{L}\right) \sin\left(\frac{k\pi z'}{L}\right) \exp\left[\frac{-i\Delta r}{2k_o}\left(\frac{k\pi}{L}\right)^2\right].\end{aligned} \tag{7.11}$$

Substituting Eq. (7.11) into Eq. (7.2) and interchanging the integration over z' with the sum over k, we get

$$\begin{aligned}\Psi(r + \Delta r, z | \epsilon, z_o) = \exp&\left\{\frac{i\Delta r k_o}{2}[n^2(r, z) - 1]\right\}\\ &\times \left(\frac{2}{L}\right)\sum_{k=1}^{\infty} \sin\left(\frac{k\pi z}{L}\right) \exp\left[-\frac{i\Delta r}{2k_o}\left(\frac{k\pi}{L}\right)^2\right]\\ &\times \int_0^L dz' \sin\left(\frac{k\pi z'}{L}\right) \Psi(r, z' | \epsilon, z_0).\end{aligned} \tag{7.12}$$

When one assumes the field vanishes at $z = L$, one introduces undesirable reflections off the artificial bottom. They are eliminated by adopting the point of view that, in water with a depth excess, the bottom acts as a perfect absorber and any acoustic energy which strikes the bottom never gets back up into the water column. The simplest way of mathematically realizing this notion is to add an imaginary term to the index of refraction:

$$n^2(r, z) \rightarrow n^2(r, z) + i\xi(z), \tag{7.13}$$

where ξ is small until z is close to L. It is typically taken to be a Gaussian:

$$\xi(z) = \frac{2\rho}{k_o}\exp\left[-\left(\frac{L - z}{L\sigma}\right)^2\right]. \tag{7.14}$$

Therefore

$$\exp\left\{\frac{i\Delta r k_o}{2}\,[n^2(r,z)\ -1]\right\}$$

$$\rightarrow \exp\left[-\Delta r\rho \exp\ -\left\{\frac{L\ -z}{L\sigma}\right\}^2\right]\exp\left\{\frac{i\Delta r k_o}{2}\,[n^2(r,z)\ -1]\right\}. \tag{7.15}$$

Since the inclusion of ξ is phenomenological, the parameters ρ and σ can be determined only through experience gained by examining the numerical results for various test cases.

The next step is to approximate the integral over z' by a discrete sum. It is assumed that over a depth increment Δz the field Ψ may be taken to be constant. Let N be the number of increments: $N\Delta z\ = L$. Then

$$\int_0^L dz' \rightarrow \Delta z \sum_{m=1}^{N},$$

and Eq. (7.12) takes the form

$$\begin{aligned}\Psi(r+\Delta r, j\Delta z|\epsilon, z_0)\ &= \exp\left[-\Delta r\rho \exp\ -\left\{\frac{N\ -j}{N\sigma}\right\}^2\right]\exp\left\{\frac{i\Delta r k_o}{2}\,[n^2(r,j\Delta z)\ -1]\right\}\\ &\times\left\{\frac{2}{N}\right\}\sum_{k=1}^{N}\sin\left\{\frac{k\pi j}{N}\right\}\exp\left[\frac{-i\Delta r}{2k_o}\left\{\frac{k\pi}{N\Delta z}\right\}^2\right]\\ &\times\sum_{m=1}^{N}\sin\left\{\frac{k\pi m}{N}\right\}\Psi(r, m\Delta z|\epsilon, z_0).\end{aligned} \tag{7.16}$$

In writing Eq. (7.16) we have used Eq. (7.15) and have truncated the sum over k. This truncated is consistent with the assumption that the wiggles in Ψ of spatial size less than ΔZ are unimportant. It is analogous to the usual Fourier sampling theorem. Although the sums in Eq. (7.16) go from 1 to N, we could sum from 0 to $N - 1$ or from 1 to $N - 1$ without changing the results.

So far we have assumed one is interested in calculating the Green's function Ψ, in which case $\Psi(\epsilon,\ m\Delta z|\epsilon,\ z_0) = \delta_{mm_0}/\Delta z$, where $z_0 = m_o\Delta z$. If one is interested in calculating the solution $\psi(r,\ z)$ to the parabolic equation corresponding to some initial distribution h,

$$\psi(\epsilon, z)\ = h(z),$$

then Eq. (7.16) is replaced by

$$\begin{aligned}\psi(r+\Delta r, j\Delta z)\ &= \exp\left[-\Delta r\rho \exp\ -\left\{\frac{N\ -j}{N\sigma}\right\}^2\right]\exp\left\{\frac{i\Delta r k_o}{2}\,[n^2(r,j\Delta z)\ -1]\right\}\\ &\times\left\{\frac{2}{N}\right\}\sum_{k=1}^{N}\sin\left\{\frac{k\pi j}{N}\right\}\exp\left[\frac{-i\Delta r}{2k_o}\left\{\frac{k\pi}{N\Delta z}\right\}^2\right]\\ &\times\sum_{m=1}^{N}\sin\left\{\frac{k\pi m}{N}\right\}\psi(r, m\Delta z),\end{aligned} \tag{7.17}$$

where

$$\psi(\epsilon, m\Delta z)\ = h(m\Delta z). \tag{7.18}$$

Let us examine the structure of Eq. (7.17) by first simplifying the notation. We define, for $j = 0, ..., N$,

$$\psi(r + \Delta r, j\Delta z) = \psi_j(r + \Delta r), \tag{7.19a}$$

$$\psi(r, j\Delta z) = \psi_j(r), \tag{7.19b}$$

$$h(j\Delta z) = h_j, \tag{7.19c}$$

$$\exp\left[-\Delta r\rho \exp -\left(\frac{N-j}{N\sigma}\right)^2\right] = \mathcal{A}_j, \tag{7.19d}$$

$$\exp\left\{\frac{i\Delta r k_o}{2}\left[n^2(r, j\Delta z) - 1\right]\right\} = \mathcal{E}_j(r), \tag{7.19e}$$

and

$$\left(\frac{2}{N}\right)\exp\left[\frac{-i\Delta r}{2k_o}\left(\frac{j\pi}{N\Delta z}\right)^2\right] = \mathcal{P}_j. \tag{7.19f}$$

Therefore

$$\psi_j(r + \Delta r) = \mathcal{A}_j\mathcal{E}_j(r)\sum_{k=0}^{N-1}\sin\left(\frac{\pi kj}{N}\right)\mathcal{P}_k\sum_{m=0}^{N-1}\sin\left(\frac{\pi km}{N}\right)\psi_m(r). \tag{7.20}$$

This equation defines the split-step Fourier algorithm. It is easy to see from Eq. (7.20) how the programming should go. To calculate the field at $r + \Delta r$, one does a sine transform on the field at r. The result is then multiplied by $\mathcal{P}_k$, and a second sine transform is performed. The output of this second transform is multiplied by $\mathcal{A}_j$ times $\mathcal{E}_j(r)$, giving $\psi_j(r + \Delta r)$. The pressure is obtained by multiplying $\psi_j(r + \Delta r)$ by some constant times $\bar{r}^{1/2}\exp ik_o(r + \Delta r)$ (see Eq. (3.4)). After the value of the pressure is transferred to a storage location, the process is repeated until the maximum range of interest has been reached. One initializes by taking

$$\psi_m(\epsilon) = h_m. \tag{7.21}$$

The matrices $\mathcal{A}_j$ and $\mathcal{P}_j$ are calculated once and stored. The matrix $\mathcal{E}_j(r)$ depends on the variability in range and depth of the sound speed and is therefore recomputed at each range step.

If one does not have a subroutine for doing a fast-Fourier sine transform, the usual exponential FFT may be used by doubling the space. The expression

$$2i\sum_{k=0}^{N-1}\sin\left(\frac{\pi kj}{N}\right)f_k = \sum_{k=0}^{2N-1}e^{2\pi ikj/2N}f_k^{dble} \tag{7.22}$$

is used, where f^{dble} is defined by the relations

$$f_0^{dble} = f_0 = 0,$$

$$f_N^{dble} = f_N = 0,$$

$$f_k^{dble} = f_k, \quad k = 1, ..., N-1,$$

and

$$f_k^{dble} = -f_{2N-k}, \quad k = N+1, ..., 2N-1. \tag{7.23}$$

Let us now try to isolate the various assumptions which went into the derivation of the split-step Fourier algorithm:

- The parabolic approximation is valid. This assumption will be discussed in Section 9.

- The density is constant. This is a valid assumption provided propagation through the bottom is unimportant.

- The step size Δr can be chosen small enough so that Eq. (7.3) is valid. As long as the sound speed is a smooth function of depth and range, a Δr can always be found. If the sound speed possesses discontinuities, then Eq. (7.3) is questionable, regardless of the size of Δr (and Δz). The reason is rather easy to see physically. Equation (7.3) means that the ray paths follow straight lines from r to $r + \Delta r$. If the sound speed is discontinuous, then, in addition to (or instead of) the straight line path, ray paths which are reflected and refracted by the discontinuity will contribute to the field at $r + \Delta r$. These reflected and refracted paths are not included in Eq. (7.3). The problem obviously persists as $\Delta z \to 0$.

- In water with a depth excess, acoustic energy which interacts with the bottom is absorbed or scattered and does not contribute to the received signal. There is some experimental evidence in favor of this assumption [77]. For a source close to the surface it will introduce some bias in the calculated pressure in the regions between convergence zones but probably not in the neighborhood of a convergence zone, where *RR* and *RSR* paths are so much more important than *BRSR* paths.

- The preceding assumption may be mathematically modeled according to Eqs. (7.5b), (7.13), and (7.14). One technique is probably as good as another. There is a situation, however, where care should be taken. If one is numerically investigating the validity of the parabolic approximation (or some improved approximation) by comparing the output of a *PE* program with that of a normal-mode program, one may be considering differences of just a few decibels. In such a situation one must be sure the absorbing bottom is modeled identically in both programs. Otherwise the results may not really say much about the validity of the parabolic approximation, particularly in the regions between convergence zones. As an example, one would a priori question the conclusions of a comparison of the field calculated using Eq. (7.20) with that calculated using a normal-mode program if the absorbing bottom is modeled in the normal-mode program by cutting off the modal sum when the phase speed exceeds the sound speed at the bottom. It would be preferable to compare the output of a program based on Eq. (7.20) with a modal solution where the normal-mode eigenfunctions $Z_n(z)$ are calculated using

$$\left\{\frac{d^2}{dz^2} + k_o^2\,[n^2(z) + i\xi(z)]\right\} Z_n(z) = k_n^2 Z_n(z) \tag{7.24}$$

with

$$Z_n(0) = Z_n(L) = 0. \tag{7.25}$$

- A reasonable value for Δz may be found. The choice of the number of depth points $N = L/\Delta z$ is intimately related to several of the other assumptions in a way which we would now like to discuss. In Eq. (7.2) we assumed Δr is small enough so that $\Psi(r + \Delta r, z|r, z')$ acts

almost like a δ-function in $z - z'$. With this assumption we were able to take the index of refraction to be a constant and hence find an analytic expression for $\Psi(r + \Delta r, z|r, z')$. The crucial point is how $\Psi(r + \Delta r, z|r, z')$ imitates the behavior of a δ-function. If $|z - z'|$ is large, Ψ will be a rapidly oscillating function of z'. Hence, when one integrates over z', cancellations will occur for large $|z - z'|$ which effectively restrict the range of integration to a small region about $z' = z$. If one does not sample often enough in depth, the oscillations are washed out, and these essential cancellations do not occur. The result is an erroneous value for the field at $r + \Delta r$.

On the other hand, one condition for the validity of the parabolic approximation is that the slopes of the Feynman paths should be small [5]: $|(z - z')/\Delta r| \ll 1$. That is, the contribution to the field from all paths with $|(z - z')/\Delta r| > 1$ should be negligible. This small-slope requirement is really not incorporated into the numerical algorithm. If the algorithm could be modified so that one keeps only those contributions to the field which correspond to $|(z - z')/\Delta r| \ll 1$, then the end result would be the same (if the parabolic approximation is valid). More importantly, one would not have to sample as often in depth, because the oscillations which occur for $|z - z'|$ large would no longer be as important. Consequently, one would conceivably save storage space and running time.

7.2. Bottom-Limited Water and Shallow Water: An Unsolved Problem

Perhaps the most important unsolved problem involving the application of the parabolic equation to sound propagation is the development of an algorithm for obtaining its solution in a bottom-limited or shallow-water situation. We will not solve this problem here but rather indicate how it might be approached.

For concreteness we will consider a specific model for the acoustic medium. We will assume it consists of two fluids in contact at the bottom interface. The depth of the water is $B(r)$, and the thickness of the bottom fluid is $L - B(r)$, with the acoustic pressure being essentially zero for depths greater than L. The density of the water is ρ_w, and the density of the bottom fluid is ρ_b. Both ρ_w and ρ_b are constant. For the index of refraction we have

$$\begin{aligned} n^2(r,z) &= n_w^2(r,z), \quad 0 < z < B(r), \\ &= n_b^2(r,z) + i\eta_b(r,z), \quad B(r) < z < L. \end{aligned} \tag{7.26}$$

where η_b represents the effect of bottom absorption.

If the water depth and index of refraction are independent of range, the pressure is given by a normal-mode sum

$$p(\mathbf{x}) = \frac{i}{4}\frac{1}{\rho(z_s)}\sum_n Z_n(z)Z_n(z_s)H_o^{(1)}(k_nR), \tag{7.27}$$

where

$$\begin{aligned} \rho(z) &= \rho_w, \quad 0 < z < B, \\ &= \rho_b, \quad B < z < L, \end{aligned} \tag{7.28}$$

and the normal-mode eigenfunctions satisfy the differential equation

$$\left[\frac{d^2}{dz^2} + k_o^2 n^2(z)\right] Z_n(z) = k_n^2 Z_n(z), \tag{7.29}$$

the boundary and continuity conditions

$$Z_n(0) = Z_n(L) = 0, \tag{7.30a}$$

$$Z_n(z) \text{ continuous}, \tag{7.30b}$$

and

$$\frac{1}{\rho(z)} \frac{d}{dz} Z_n(z) \text{ continuous}, \tag{7.30c}$$

and the orthonormality condition

$$\int_0^L \frac{dz}{\rho(z)} Z_n(z) Z_m(z) = \delta_{nm}. \tag{7.31}$$

These functions are not normalized in the conventional way because we are assuming the pressure, rather than the particle velocity potential, satisfies the Helmholtz equation.

In Section 9 we will show the parabolic approximation is equivalent to replacing Eq. (7.27) by

$$p(\mathbf{x}) = \frac{1}{4\pi} \left[\frac{2\pi i}{k_o R}\right]^{1/2} e^{ik_o R} \Psi(R, z|0, z_s) \tag{7.32}$$

with

$$\Psi(R, z|0, z_s) = \frac{1}{\rho(z_s)} \sum_n Z_n(z) Z_n(z_s) \exp\left[\frac{iR}{2k_o}(k_n^2 - k_o^2)\right]. \tag{7.33}$$

By use of the orthonormality condition (Eq. (7.31)), it is easy to show that Ψ obeys the composition law:

$$\Psi(r + \Delta r, z|0, z_s) = \int_0^L dz' \Psi(r + \Delta r, z|r, z') \Psi(r, z'|0, z_s), \tag{7.34}$$

where

$$\Psi(r + \Delta r, z|r, z') = \frac{1}{\rho(z')} \sum_n Z_n(z) Z_n(z') \exp\left[\frac{i\Delta r}{2k_o}(k_n^2 - k_o^2)\right] \tag{7.35}$$

With Eqs. (7.34) and (7.35) we have a marching algorithm for solving the parabolic equation for the range-independent situation. Of course one would not use these equations in practice since the pressure is readily obtained from Eq. (7.27). We have derived them to fix the notation and because their structure suggests how one might generalize to the range-dependent problem.

As we have mentioned, the problem of constructing an algorithm in the general case reduces to the problem of determining an expression for $\Psi(r + \Delta r, z|r, z')$. If one has such an expression, the composition law can be used to construct the field at $r + \Delta r$ in terms of the field at r. If the parabolic approximation is to be at all valid, a value of Δr can be found such that the variation with range of the medium's acoustic parameters can be ignored within the region r to $r + \Delta r$. With this approximation $\Psi(r + \Delta r, z|r, z')$ has the form of Eq. (7.35) but with the usual normal modes replaced by range-dependent normal modes $Z_n(r, z)$ [78]. These range-dependent eigenfunctions and their corresponding eigenvalues are obtained by solving an eigenvalue problem at the particular range r.

Specifically we have

$$\Psi(r + \Delta r, z | r, z') = \frac{1}{\rho(z')} \sum_n Z_n(r, z) Z_n(r, z') \exp\left\{\frac{i\Delta r}{2k_o} [k_n^2(r) - k_o^2]\right\}, \tag{7.36}$$

where

$$\begin{aligned} Z_n(r, z) &= Z_n^w(r, z), \quad 0 < z < B(r), \\ &= Z_n^b(r, z), \quad B(r) < z < L, \end{aligned} \tag{7.37}$$

with

$$\left[\frac{\partial^2}{\partial z^2} + k_o^2 n_w^2(r, z)\right] Z_n^w(r, z) = k_n^2(r) Z_n^w(r, z) \tag{7.38a}$$

for $0 < z < B(r)$ and

$$\left\{\frac{\partial^2}{\partial z^2} + k_o^2 [n_p^2(r, z) + i\eta_b(r, z)]\right\} Z_n^b(r, z) = k_n^2(r) Z_n^b(r, z) \tag{7.38b}$$

for $B(r) < z < L$. The boundary and continuity conditions are

$$Z_n^w(r, 0) = Z_n^b(r, L) = 0, \tag{7.39a}$$

$$Z_n^w(r, B(r)) = Z_n^b(r, B(r)), \tag{7.39b}$$

and

$$\frac{1}{\rho_w} \frac{\partial}{\partial z} Z_n^w(r, z)\Big|_{z=B(r)} = \frac{1}{\rho_b} \frac{\partial}{\partial z} Z_n^b(r, z)\Big|_{z=B(r)}. \tag{7.39c}$$

If we further define

$$\begin{aligned} \Psi(r, z | o, z_s) &= \Psi^w(r, z), \quad 0 < z < B(r), \\ &= \Psi^b(r, z), \quad B(r) < z < L, \end{aligned} \tag{7.40}$$

then the pressure at a point (R, z) in the water column is

$$p(\mathbf{x}) = \frac{1}{4\pi} \left(\frac{2\pi i}{k_o R}\right)^{1/2} e^{ik_o R} \Psi^w(R, z),$$

where Ψ is obtained by using the algorithm

$$\begin{aligned} \Psi^w(r + \Delta r, z) = e^{-i\Delta r k_o/2} \sum_n Z_n^w(r, z) e^{i\Delta r k_n^2(r)/2k_o} \\ \times \left\{\frac{1}{\rho_w} \int_0^{B(r)} dz' \, Z_n^w(r, z') \Psi^w(r, z') \right. \\ \left. + \frac{1}{\rho_b} \int_{B(r)}^{L} dz' \, Z_n^b(r, z') \Psi^b(r, z') \right\} \end{aligned} \tag{7.41a}$$

and

$$\Psi^b(r+\Delta r, z) = e^{-i\Delta r k_o/2} \sum_n Z_n^b(r,z) e^{i\Delta r k_n^2(r)/2k_o} \times \left\{ \frac{1}{\rho_w} \int_0^{B(r)} dz'\, Z_n^w(r,z')\Psi^w(r,z') + \frac{1}{\rho_b} \int_{B(r)}^{L} dz'\, Z_n^b(r,z')\Psi^b(r,z') \right\}. \tag{7.41b}$$

Since this approach has never been tested, it is not appropriate to discuss the algorithm, Eqs. (7.41), in detail. We would like to make two points however. First, we have ignored any contribution due to shear waves. It is possible in principle to generalize the whole formalism to include shear waves in the same way quantum mechanics is generalized to include particle polarization (spin).

Second, range-dependent normal modes have been used [78] to solve the Helmholtz equation. The main problem with their use is that the coupled differential equations for the modal coefficients are extremely difficult to solve. Equations (7.41) really amount to solving these equations in the parabolic approximation. However Eqs. (7.41) represent an unsophisticated, brute-force technique for solving the parabolic equation. A complete set of normal modes mist be calculated for each range step. One imagines there is a simpler, more sophisticated technique than one based on range-dependent normal modes.

8. PATH INTEGRATION AND THE HELMHOLTZ EQUATION

In this section we will develop path-integral representations for the solution to the Helmholtz equation and consider a few simple applications. We shall assume the medium is unbounded. It will be apparent that many of our results are valid even for a bounded medium.

8.1. Some Basic Representations

We begin with the Helmholtz equation, Eq. (3.2):

$$[\nabla^2 + k_o^2 n^2(\mathbf{x})]p(\mathbf{x}) = -\delta^{(3)}(\mathbf{x}-\mathbf{x}_s), \tag{8.1}$$

where $k_o = \omega/c_o$, $n(\mathbf{x}) = c_o/c(\mathbf{x})$, and we have suppressed any time dependence. With an integration by parts, one can show

$$p(\mathbf{x}) = \frac{i}{2k_o} \int_0^\infty d\tau\, e^{i\tau(k_o + i\epsilon)/2} \Phi(\tau, \mathbf{x}|0, \mathbf{x}_s), \tag{8.2}$$

where the propagator Φ satisfies the equations

$$-2ik_o\partial_\tau \Phi(\tau,\mathbf{x}|\tau',\mathbf{x}') = \{\nabla^2 + k_o^2[n^2(\mathbf{x}) - 1]\}\Phi(\tau,\mathbf{x}|\tau',\mathbf{x}') \tag{8.3a}$$

and

$$\Phi(\tau',\mathbf{x}|\tau',\mathbf{x}') = \delta^{(3)}(\mathbf{x}-\mathbf{x}'), \text{ for all } \tau'. \tag{8.3b}$$

As before, it will be convenient to define

$$U(\mathbf{x}) = -\frac{1}{2}[n^2(\mathbf{x}) - 1]. \tag{8.4}$$

Formally, at least, we have the composition law

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \int d\mathbf{x}' \, \Phi(\tau, \mathbf{x}|\sigma, \mathbf{x}') \Phi(\sigma, \mathbf{x}'|0, \mathbf{x}_s), \tag{8.5}$$

where σ is specified only by the condition $\tau \geqslant \sigma \geqslant 0$. If we iterate Eq. (8.5), we obtain

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \int d\mathbf{x}_1 \ldots d\mathbf{x}_{N-1} \prod_{i=1}^{N} \Phi(\sigma_i, \mathbf{x}_i|\sigma_{i-1}, \mathbf{x}_{i-1}), \tag{8.6}$$

with $\sigma_0 = 0$, $\mathbf{x}_0 = \mathbf{x}_s$, $\sigma_N = \tau$, and $\mathbf{x}_N = \mathbf{x}$. We will choose increments of equal length,

$$\sigma_i - \sigma_{i-1} = \Delta\sigma, \tag{8.7}$$

for all i. We now assume N is large enough ($\Delta\sigma$ is small enough) so that $U(\mathbf{x})$ can be considered constant when calculating each of the component propagators in Eq. (8.6). We therefore have

$$\Phi(\sigma_i, \mathbf{x}_i|\sigma_{i-1}, \mathbf{x}_{i-1}), \simeq e^{-ik_o\Delta\sigma U(\mathbf{x}_i)} \Phi_o(\sigma_i, \mathbf{x}_i|\sigma_{i-1}, \mathbf{x}_{i-1}), \tag{8.8}$$

where Φ_o satisfies the equations

$$-2ik_o\partial_{\sigma_i}\Phi_o(\sigma_i, \mathbf{x}_i|\sigma_{i-1}, \mathbf{x}_{i-1}) = \nabla_i^2\Phi_o(\sigma_i, \mathbf{x}_i|\sigma_{i-1}, \mathbf{x}_{i-1}) \tag{8.9a}$$

and

$$\Phi_o(\sigma_{i-1}, \mathbf{x}_i|\sigma_{i-1}, \mathbf{x}_{i-1}) = \delta^{(3)}(\mathbf{x}_i - \mathbf{x}_{i-1}). \tag{8.9b}$$

The solution to Eqs. (8.9) is

$$\Phi_o(\sigma_i, \mathbf{x}_i|\sigma_{i-1}, \mathbf{x}_{i-1}) = \left(\frac{k_o}{2\pi i\Delta\sigma}\right)^{3/2} \exp\left[\frac{ik_o\Delta\sigma}{2}\left(\frac{\mathbf{x}_i - \mathbf{x}_{i-1}}{\Delta\sigma}\right)^2\right]. \tag{8.10}$$

Gathering expressions gives

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) \simeq \left(\frac{k_o}{2\pi i\Delta\sigma}\right)^{3N/2} \int d\mathbf{x}_1 \ldots d\mathbf{x}_{N-1}$$

$$\times \exp\left\{ik_o\Delta\sigma \sum_{i=1}^{N}\left[\frac{1}{2}\left(\frac{\mathbf{x}_i - \mathbf{x}_{i-1}}{\Delta\sigma}\right)^2 - U(\mathbf{x}_i)\right]\right\}. \tag{8.11}$$

In the continuum limit, $N \rightarrow \infty$, with $N\Delta\sigma = \tau$ fixed, Eq. (8.11) becomes exact. We have

$$\{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_N\} \rightarrow \mathbf{x}(\sigma), \quad 0 \leqslant \sigma \leqslant \tau, \tag{8.12a}$$

$$\left\{\frac{\mathbf{x}_1 - \mathbf{x}_0}{\Delta\sigma}, \frac{\mathbf{x}_2 - \mathbf{x}_1}{\Delta\sigma}, \ldots, \frac{\mathbf{x}_N - \mathbf{x}_{N-1}}{\Delta\sigma}\right\} \rightarrow \frac{d\mathbf{x}(\sigma)}{d\sigma}, \tag{8.12b}$$

$$\Delta\sigma \sum_{i=1}^{N} \rightarrow \int_0^\tau d\sigma, \tag{8.12c}$$

and

$$\left(\frac{k_o}{2\pi i\Delta\sigma}\right)^{3N/2} \int d\mathbf{x}_1 \ldots d\mathbf{x}_{N-1} \rightarrow \int D[\mathbf{x}(\sigma)], \tag{8.12d}$$

giving

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \int D[\mathbf{x}(\sigma)] \exp\left\{ik_o \int_0^\tau d\sigma \left[\frac{1}{2}\left(\frac{d\mathbf{x}}{d\sigma}\right)^2 - U(\mathbf{x}(\sigma))\right]\right\}. \tag{8.13}$$

According to Eq. (8.12a) the paths $\mathbf{x}(\sigma)$ in Eq. (8.13) satisfy the endpoint conditions

$$\mathbf{x}(0) = \mathbf{x}_s, \quad \mathbf{x}(\tau) = \mathbf{x}. \tag{8.14}$$

Substituting Eq. (8.13) into Eq. (8.2) gives

$$p(\mathbf{x}) = \frac{i}{2k_o} \int_0^\infty d\tau\, e^{i\tau(k_o + i\epsilon)/2} \int D[\mathbf{x}(r)] \exp\left\{ik_o \int_0^\tau d\sigma \left[\frac{1}{2}\left(\frac{d\mathbf{x}}{d\sigma}\right)^2 - U(\mathbf{x}(\sigma))\right]\right\}. \tag{8.15}$$

Because the measure depends on τ, the integral over τ cannot be interchanged with the integral over the paths. Using Eq. (8.4) we can rewrite Eq. (8.15) in the form

$$p(\mathbf{x}) = \frac{i}{2k_o} \int_0^\infty d\tau \int D[\mathbf{x}(\sigma)] \exp\left\{\frac{ik_o}{2} \int_0^\tau d\sigma \left[\left(\frac{d\mathbf{x}}{d\sigma}\right)^2 + n^2(\mathbf{x}(\sigma))\right]\right\}, \tag{8.16}$$

where again the integration is over all continuous paths satisfying the endpoint conditions, Eq. (8.14). This is our first path-integral representation for the solution to the Helmholtz equation.

Just as with the parabolic equation, it is possible to introduce a change of variable

$$\mathbf{v}_i = \frac{\mathbf{x}_i - \mathbf{x}_{i-1}}{\Delta\sigma}, \quad i = 1, \ldots, N, \tag{8.17}$$

and construct the velocity representation. Taking

$$\left(\frac{k_o \Delta\sigma}{2\pi i}\right)^{3N/2} \int d\mathbf{v}_1 \ldots d\mathbf{v}_N \rightarrow \int D[\mathbf{v}(\sigma)], \tag{8.18}$$

we have

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \int D[\mathbf{v}(\sigma)]\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_s - \int_0^\tau d\sigma\, \mathbf{v}(\sigma)\right)$$
$$\times \exp\left\{ik_o \int_0^\tau d\sigma \left[\frac{1}{2}\mathbf{v}^2(\sigma) - U\left(\mathbf{x}_s + \int_0^\sigma d\sigma'\, \mathbf{v}(\sigma')\right)\right]\right\}. \tag{8.19}$$

We therefore have our second representation for the pressure field:

$$p(\mathbf{x}) = \frac{i}{2k_o} \int_0^\infty d\tau e^{i\tau(k_o + i\epsilon)/2} \int D[\mathbf{v}(\sigma)]\, \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_s - \int_0^\tau d\sigma \mathbf{v}(\sigma)\right)$$
$$\times \exp\left\{ik_o \int_0^\tau d\sigma \left[\frac{1}{2}\mathbf{v}^2(\sigma) - U\left(\mathbf{x}_s + \int_0^\sigma d\sigma' \mathbf{v}(\sigma')\right)\right]\right\}. \tag{8.20}$$

or

$$p(\mathbf{x}) = \frac{i}{2k_o} \int_0^\infty d\tau \int D[\mathbf{v}(\sigma)] \delta^{(3)} \left[\mathbf{x} - \mathbf{x}_s - \int_0^\tau d\sigma \mathbf{v}(\sigma) \right]$$

$$\times \exp \left\{ \frac{ik_o}{2} \int_0^\tau d\sigma \left[\mathbf{v}^2(\sigma) + n^2 \left(\mathbf{x}_s + \int_0^\sigma d\sigma' \mathbf{v}(\sigma') \right) \right] \right\}. \quad (8.21)$$

Apparently this particular representation was first obtained by Fradkin [12].

We shall construct a third representation by introducing a new computational technique. Basically we want to redefine the path $\mathbf{x}(\sigma)$, $0 \leqslant \sigma \leqslant \tau$, according to the transformation

$$\mathbf{w}(r) = \sqrt{\frac{R'}{\tau}}\, \mathbf{x}\left(\frac{\tau}{R'} r \right) \quad (8.22)$$

and integrate over $\mathbf{w}$ rather than $\mathbf{x}$. Here R' will be a free parameter. When we consider applications, we will let R' be the range. The advantage of this transformation is that the resulting path integral will have a form similar to that of the path integral for the parabolic equation.

The path in Eq. (8.22) is defined on the interval $0 \leqslant r \leqslant R'$ and satisfies the endpoint conditions

$$\mathbf{w}(0) = \sqrt{\frac{R'}{\tau}}\, \mathbf{x}_s, \quad \mathbf{w}(R') = \sqrt{\frac{R'}{\tau}}\, \mathbf{x}. \quad (8.23)$$

For the exponent in Eq. (8.13) we have

$$\int_0^\tau d\sigma \left[\frac{1}{2} \left(\frac{d\mathbf{x}}{d\sigma} \right)^2 - U(\mathbf{x}(\sigma)) \right] = \int_0^{R'} dr \left[\frac{1}{2} \left(\frac{d\mathbf{w}}{dr} \right)^2 - \frac{\tau}{R'} U\left(\sqrt{\frac{\tau}{R'}}\, \mathbf{w}(r) \right) \right]. \quad (8.24)$$

For the measure we have

$$\int D[\mathbf{x}(\sigma)] \sim \left(\frac{k_o}{2\pi i \Delta\sigma} \right)^{3N/2} \int d\mathbf{x}_1 \ldots d\mathbf{x}_{N-1}$$

$$\sim \left(\frac{k_o}{2\pi i \Delta\sigma} \right)^{3N/2} \int d\mathbf{x}(\sigma_1) \ldots d\mathbf{x}(\sigma_{N-1})$$

$$\sim \left(\frac{k_o R'}{2\pi i \tau \Delta r} \right)^{3N/2} \int d\mathbf{x}\left(\frac{\tau}{R'} r_1 \right) \ldots d\mathbf{x}\left(\frac{\tau}{R'} r_{N-1} \right). \quad (8.25)$$

Thus

$$\int D[\mathbf{x}(\sigma)] = \left(\frac{R'}{\tau} \right)^{3/2} \int D[\mathbf{w}(r)], \quad (8.26)$$

where

$$\left(\frac{k_o}{2\pi i \Delta r} \right)^{3N/2} \int d\mathbf{w}_1 \ldots d\mathbf{w}_{N-1} \to \int D[\mathbf{w}(r)]. \quad (8.27)$$

The path integral in Eq. (8.13) becomes

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) \equiv \left(\frac{R'}{\tau}\right)^{3/2} \Phi_{\tau/R'}(R', \mathbf{x}|0, \mathbf{x}_s), \tag{8.28}$$

where

$$\Phi_{\tau/R'}(R', \mathbf{x}|0, \mathbf{x}_s) = \int D[\mathbf{w}(r)] \exp\left\{ik_o \int_0^{R'} dr \left[\frac{1}{2}\left(\frac{d\mathbf{w}}{dr}\right)^2 - \frac{\tau}{R'} U\left(\sqrt{\frac{\tau}{R'}}\,\mathbf{w}(r)\right)\right]\right\}, \tag{8.29}$$

with $\mathbf{w}(0) = \sqrt{(R'/\tau)}\,\mathbf{x}_s$ and $\mathbf{w}(R') = \sqrt{(R'/\tau)}\,\mathbf{x}$. Before substituting into the integral over τ which determines the pressure, we let $\beta = \tau/R'$. Hence

$$p(\mathbf{x}) = \frac{iR'}{2k_o} \int_0^\infty \frac{d\beta}{\beta^{3/2}}\, e^{ik_o R'\beta/2}\, \Phi_\beta(R', \mathbf{x}|0, \mathbf{x}_s). \tag{8.30}$$

(It is implicitly assumed that k_o has a small, positive imaginery part.) This is our third representation and the last one we will derive.

8.2. A Homogeneous Medium

If the medium is homogeneous so that $n(\mathbf{x})$ is constant ($= n_o$), the complicated path-integral representations of the previous subsection must give

$$p(\mathbf{x}) = \frac{e^{ik_o n_o|\mathbf{x} - \mathbf{x}_s|}}{4\pi|\mathbf{x} - \mathbf{x}_s|}. \tag{8.31}$$

Let us verify that this is indeed the case for the representation Eqs. (8.29) and (8.30). It is not difficult to show the path integral in Eq. (8.29) is normalized so that

$$\int D[\mathbf{w}(r)] \exp\left\{\frac{ik_o}{2}\int_0^{R'} dr \left(\frac{d\mathbf{w}}{dr}\right)^2\right\} = \left(\frac{k_o}{2\pi i R'}\right)^{3/2} \exp\left\{\frac{ik_o R'}{2}\left[\frac{\mathbf{w}(R') - \mathbf{w}(0)}{R'}\right]^2\right\}$$

$$= \left(\frac{k_o}{2\pi i R'}\right)^{3/2} \exp\left[\frac{ik_o R'}{2\beta}\left(\frac{\mathbf{x} - \mathbf{x}_s}{R'}\right)^2\right]. \tag{8.32}$$

If the index of refraction is independent of **x**, then

$$\int_0^{R'} dr\left[-\beta U(\sqrt{\beta}\,\mathbf{w}(r))\right] = -R'\beta U = +\frac{1}{2}R'\beta(n_o^2 - 1). \tag{8.33}$$

We therefore have

$$\Phi_\beta(R', \mathbf{x}|0, \mathbf{x}_s) = \left(\frac{k_o}{2\pi i R'}\right)^{3/2} \exp\left\{\frac{ik_o R'}{2}\left[\beta n_o^2 - \beta + \frac{1}{\beta}\left(\frac{\mathbf{x} - \mathbf{x}_s}{R'}\right)^2\right]\right\}. \tag{8.34}$$

Substituting this into Eq. (8.30) yields

$$p(\mathbf{x}) = \frac{1}{4\pi}\left(\frac{k_o}{2\pi i R'}\right)^{1/2}\int_0^\infty \frac{d\beta}{\beta^{3/2}}\exp\left\{\frac{ik_oR'}{2}\left[\beta n_o^2 + \frac{1}{\beta}\left(\frac{\mathbf{x}-\mathbf{x}_s}{R'}\right)^2\right]\right\}. \tag{8.35}$$

By letting $\tau = n_o R'\beta/|\mathbf{x} - \mathbf{x}_s|$, this equation becomes

$$p(\mathbf{x}) = \frac{1}{4\pi}\left(\frac{k_o n_o}{2\pi i|\mathbf{x}-\mathbf{x}_s|}\right)^{1/2}\int_0^\infty \frac{d\tau}{\tau^{3/2}}\exp\left[\frac{ik_o n_o|\mathbf{x}-\mathbf{x}_s|}{2}\left(\tau + \frac{1}{\tau}\right)\right]. \tag{8.36}$$

In Appendix A a one-parameter integral representation for the free-field Green's function in n dimensions is constructed. By comparing Eq. (8.36) with Eq. (A13), we obtain the desired result, Eq. (8.31). Notice that R' canceled out, as it should, since Eq. (8.30) is independent of R'.

8.3. The Parabolic Profile

No general discussion of Feynman path integrals would be complete without mention of the parabolic profile model. In this subsection we shall examine this model and point out the relationship between the path-integral solution and the usual normal-mode solution.

We first observe that if U is a function of just depth, $U(\mathbf{x}) = U(z)$, the path integral in Eq. (8.29) (with $\beta = \tau/R'$) may be written as the product of three path integrals:

$$\Phi_\beta(R', \mathbf{x}|0, \mathbf{x}_s) = \Phi_x\Phi_y\Phi_z, \tag{8.37}$$

where

$$\Phi_x = \int D[w_x(r)]\exp\left[\frac{ik_o}{2}\int_0^{R'} dr\left(\frac{dw_x}{dr}\right)^2\right], \tag{8.38a}$$

$$\Phi_y = \int D[w_y(r)]\exp\left[\frac{ik_o}{2}\int_0^{R'} dr\left(\frac{dw_y}{dr}\right)^2\right], \tag{8.38b}$$

and

$$\Phi_z = \int D[w_z(r)]\exp\left[\frac{ik_o}{2}\int_0^{R'} dr\left(\frac{dw_z}{dr}\right)^2 - 2ik_o\beta\int_0^{R'} dr U(\sqrt{\beta}w_z)\right], \tag{8.38c}$$

in which

$$w_x(0) = \beta^{-1/2}x_s, \quad w_x(R') = \beta^{-1/2}x,$$

$$w_y(0) = \beta^{-1/2}y_s, \quad w_y(R') = \beta^{-1/2}y,$$

and

$$w_z(0) = \beta^{-1/2}z_s, \quad w_z(R') = \beta^{-1/2}z. \tag{8.39}$$

The integrals are normalized so that

$$\left(\frac{k_o}{2\pi i\Delta r}\right)^{N/2}\int dw_{x_1}\ldots dw_{x_{N-1}} \to \int D[w_x(r)],$$

with similar expressions for $\int D[w_y(r)]$ and $\int D[w_z(r)]$. It should be obvious by now that

$$\Phi_x = \left[\frac{k_o}{2\pi i R'}\right]^{1/2} \exp\left[\frac{ik_oR'}{2\beta}\left[\frac{x - x_s}{R'}\right]^2\right] \tag{8.40a}$$

and

$$\Phi_y = \left[\frac{k_o}{2\pi i R'}\right]^{1/2} \exp\left[\frac{ik_oR'}{2\beta}\left[\frac{y - y_s}{R'}\right]^2\right]. \tag{8.40b}$$

Substituting these into Eq. (8.30) gives

$$p(\mathbf{x}) = \frac{1}{4\pi}\int_0^\infty \frac{d\beta}{\beta^{3/2}}\, \Phi_z \exp\left\{\frac{ik_oR'}{2}\left[\beta + \frac{1}{\beta}\left[\frac{x - x_s}{R'}\right]^2 + \frac{1}{\beta}\left[\frac{y - y_s}{R'}\right]^2\right]\right\}. \tag{8.41}$$

Although we could keep R' a free parameter and demonstrate that it cancels out in the end, it simplifies the notation to set $R' = R = [(x - x_s)^2 + (y - y_s)^2]^{1/2}$. The exponent in Eq. (8.41) then becomes

$$\frac{ik_oR}{2}\left\{\beta + \frac{1}{\beta}\right\}.$$

We let

$$n^2(z) = \left[\frac{c_o}{c(z)}\right]^2 = 1 - \alpha^2(z - z_M)^2, \tag{8.42}$$

where z_M is the depth of the SOFAR axis. Hence

$$\beta U(\sqrt{\beta}\, w_z(r)) = \frac{1}{2}\beta^2\alpha^2[w_z(r) - \beta^{-1/2}z_M]^2. \tag{8.43}$$

Therefore

$$\Phi_z = \int D[w_z(r)] \exp\left\{\frac{ik_o}{2}\int_0^r dr\left[\left[\frac{dw_z}{dr}\right]^2 - \beta^2\alpha^2[w_z(r) - \beta^{1/2}z_M]^2\right]\right\}. \tag{8.44}$$

We know that ray acoustics gives the exact value for the path integral if U is at most quadratic in the path variable. We will use this fact to evaluate Φ_z. Since Φ_z has exactly the same form as the path integral we considered in Section 3, we write

$$\Phi_z = \left[\frac{k_o}{2\pi i\, dw_z^*(R)/dp(0)}\right]^{1/2} \exp\,[ik_oA(w_z^*(r))], \tag{8.45}$$

where $p(0) = dw_z^*(r)/dr$ evaluated at $r = 0$ and

$$A(w_z^*(r)) = \frac{1}{2}\int_0^R dr\left\{\left[\frac{dw_z^*}{dr}\right]^2 - \beta^2\alpha^2[w_z^*(r) - \beta^{-1/2}z_M]^2\right\}. \tag{8.46}$$

The path $w_z^*(r)$ satisfies the differential equation

$$\frac{d^2w_z^*}{dr^2} = -\beta^2\alpha^2(w_z^* - \beta^{-1/2}z_M) \tag{8.47}$$

with $w_z^*(0) = \beta^{-1/2} z_s$ and $w_z^*(R) = \beta^{-1/2} z$. The solution to Eq. (8.47) with $w_z^*(0) = \beta^{-1/2} z_s$ and $p(0) = dw_z^*(0)/dr$ is

$$w_z^*(r) = \beta^{-1/2}[z_M + (z_s - z_M)\cos\alpha\beta r] + \frac{p(0)}{\alpha\beta}\sin\alpha\beta r. \tag{8.48}$$

Hence

$$w_z^*(R) = \beta^{-1/2}[z_M + (z_s - z_M)\cos\alpha\beta R] + \frac{p(0)}{\alpha\beta}\sin\alpha\beta R. \tag{8.49}$$

From this last equation we have

$$\frac{dw_z^*(R)}{dp(0)} = \frac{\sin\alpha\beta R}{\alpha\beta}. \tag{8.50}$$

The caustics are at $\alpha\beta R = n\pi$, $n = 1, 2, \ldots$. If Eq. (8.49) is now used to eliminate $p(0)$ from the expression for $w_z^*(r)$, it only takes some algebra to show

$$A(w_z^*(r)) = \frac{\alpha}{2\sin\alpha\beta R}\{[(z - z_M)^2 + (z_s - z_M)^2]\cos\alpha\beta R - 2(z - z_M)(z_s - z_M)\}. \tag{8.51}$$

Collecting terms gives

$$\Phi_z = \left[\frac{k_o\alpha\beta}{2\pi i\sin\alpha\beta R}\right]^{1/2}\exp\left\{\frac{ik_o\alpha}{2\sin\alpha\beta R}\left\{[(z - z_M)^2 + (z_s - z_M)^2]\cos\alpha\beta R - 2(z - z_M)(z_s - z_M)\right\}\right\}. \tag{8.52}$$

The solution to the parabolic equation for the parabolic profile is thus extremely simple. Although this solution has been known for at least 30 years, it is continuously being rediscovered. Equation (8.52) may be generalized to include the effects of a pressure-release surface by using the techniques of Ref. 15, and a solution may also be readily obtained if α has a range dependence.

Our expression for Φ_z may be written as an infinite sum of products of Hermite polynomials H_n by using the formula [79]

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{e^{-in\theta}}{2^n n!} = \left[\frac{1}{2i\sin\theta}\right]^{1/2}\exp\left\{\frac{i\theta + x^2 + y^2}{2} + \frac{i}{2\sin\theta}[(x^2 + y^2)\cos\theta - 2xy]\right\}.$$

Letting $\theta = \alpha\beta R$ gives

$$\Phi_z = \beta^{1/2}\sum_{n=0}^{\infty} Z_n(z - z_M)Z_n(z_s - z_M)\exp\left[\frac{-i\alpha\beta R}{2}(2n + 1)\right], \tag{8.53}$$

where

$$Z_n(z) = \left[\frac{k_o\alpha}{\pi}\right]^{1/4}(2^n n!)^{-1/2}H_n(\sqrt{k_o\alpha}\,z)\exp\left[-\frac{1}{2}k_o\alpha z^2\right]. \tag{8.54}$$

These functions obey the orthonormality condition

$$\int_{-\infty}^{\infty} dz\, Z_n(z) Z_m(z) = \delta_{nm}. \tag{8.55}$$

We substitute Eq. (8.53) into Eq. (8.41):

$$p(\mathbf{x}) = \sum_{n=0}^{\infty} Z_n(z - z_M) Z_n(z_s - z_M) \frac{1}{4\pi} \int_0^{\infty} \frac{d\beta}{\beta} \exp\left[\frac{ik_o R}{2}\left(\frac{1}{\beta} + \beta \frac{k_n^2}{k_o^2}\right)\right], \tag{8.56}$$

where

$$k_n^2 = k_o^2 - \alpha k_o (2n + 1). \tag{8.57}$$

By letting $\beta = k_o \tau / k_n$, the integral in Eq. (8.56) becomes

$$\frac{1}{4\pi} \int_0^{\infty} \frac{d\tau}{\tau} \exp\left[\frac{ik_n R}{2}\left(\tau + \frac{1}{\tau}\right)\right],$$

which according to Eq. (A12) is $(i/4) H_o^{(1)}(k_n R)$. Hence we obtain

$$p(\mathbf{x}) = \frac{i}{4} \sum_{n=0}^{\infty} Z_n(z - z_M) Z_n(z_s - z_M) H_o^{(1)}(k_n R), \tag{8.58}$$

which is the standard result.

8.4. Modified Perturbation Theory and the Supereikonal Approximation

We shall consider now the application of Fradkin's modified perturbation theory [12] to the Helmholtz equation. This scheme has been developed for underwater sound by Munk and Zachariasen [80] and by Callan and Zachariasen [81] by working order by order in standard perturbation theory. Modified perturbation theory gives in lowest order an approximation, coined the "supereikonal approximation" in Refs. 80 and 81, closely related to the Rytov approximation [74-76].

Returning to Eqs. (8.2) and (8.3), we define Q by

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \Phi_o(\tau, \mathbf{x}|0, \mathbf{x}_s) \exp Q, \tag{8.59}$$

where

$$\Phi_o(\tau, \mathbf{x}|\tau', \mathbf{x}') = \left[\frac{k_o}{2\pi i(\tau - \tau')}\right]^{3/2} \exp\left[\frac{ik_o(\tau - \tau')}{2}\left(\frac{\mathbf{x} - \mathbf{x}'}{\tau - \tau'}\right)^2\right]. \tag{8.60}$$

The function Φ_o is the solution to Eqs. (8.3) for $n = 1$ $(U = 0)$. Equation (8.3b) will be satisfied if $Q = 0$ at $\tau = 0$. It follows from Eq. (8.3a) that Q satisfies

$$-2ik_o \partial_\tau Q = \frac{2ik_o}{\tau} (\mathbf{x} - \mathbf{x}_s) \cdot \nabla Q + \nabla^2 Q + (\nabla Q)^2 - 2k_o^2 U(\mathbf{x}). \tag{8.61}$$

We wish to obtain a series expansion for Q in powers of the perturbation $U(\mathbf{x})$. The simplest way of obtaining this is to replace $U(\mathbf{x})$ by $gU(\mathbf{x})$, where g is an expansion parameter which is set equal to unity at the end of the calculation. We then write

$$Q = \sum_{l=1}^{\infty} g^l Q^{(l)}. \tag{8.62}$$

After substituting Eq. (8.62) into Eq. (8.61), we equate like powers of g and obtain the coefficients $Q^{(l)}$. The series in Eq. (8.62) starts with $l = 1$ because $\Phi = \Phi_o$ ($Q^{(0)} = 0$) if $U = 0$. For $l = 1$ we have

$$- 2ik_o\partial_\tau Q^{(1)} = \frac{2ik_o}{\tau}(\mathbf{x} - \mathbf{x}_s) \cdot \nabla Q^{(1)} + \nabla^2 Q^{(1)} - 2k_o^2 U. \tag{8.63}$$

The terms with $l > 1$ are obtained from the recursion relation

$$- 2ik_o\partial_\tau Q^{(l)} = \frac{2ik_o}{\tau}(\mathbf{x} - \mathbf{x}_s) \cdot \nabla Q^{(l)} + \nabla^2 Q^{(l)} + \sum_{m=1}^{l-1} \nabla Q^{(m)} \cdot \nabla Q^{(l-m)}.$$

If we could solve Eqs. (8.63) and (8.64), we would have the exact solution

$$\Phi = \Phi_o \exp\left[\sum_{l=1}^{\infty} Q^{(l)}\right]. \tag{8.65}$$

Here we will be content with the approximation

$$\Phi = \Phi_o \exp Q^{(1)}. \tag{8.66}$$

The higher-order terms have been discussed by Fradkin [12].

Equation (8.63) is not difficult to solve. One writes

$$Q' = Q^{(1)}\Phi_o(\tau, \mathbf{x}|0, \mathbf{x}_s) \tag{8.67}$$

and then solves the resulting differential equation for Q' by using standard Green's-function techniques. The result is

$$Q^{(1)} = \frac{-ik_o}{\Phi_o(\tau, \mathbf{x}|0, \mathbf{x}_s)} \int d\mathbf{x}'\, U(\mathbf{x}') \int_0^\tau d\tau'\, \Phi_o(\tau, \mathbf{x}|\tau', \mathbf{x}')\Phi_o(\tau', \mathbf{x}'|0, \mathbf{x}_s). \tag{8.68}$$

Comparison of Eqs. (8.66) and (8.68) with Eq. (6.81) shows the approximation $Q = Q^{(1)}$ is equivalent to the Rytov approximation for Φ. The preceeding analysis represents a different technique for obtaining the Rytov approximation; one based on the differential equation satisfied by Φ rather than the path integral.

The expression for $Q^{(1)}$ may be cast into a different form by introducing the Fourier transform

$$U(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} U(\mathbf{k}) \tag{8.69}$$

and carrying out the integration over $\mathbf{x}'$ in Eq. (8.68):

$$Q^{(1)} = -ik_o \int \frac{d\mathbf{k}}{(2\pi)^3} U(\mathbf{k}) \int_0^\tau d\tau' \exp\left\{i\mathbf{k} \cdot \left[\mathbf{x}_s + \frac{\tau'}{\tau}(\mathbf{x} - \mathbf{x}_s)\right] - \frac{i\mathbf{k}^2}{2k_o}\tau'\left[1 - \frac{\tau'}{\tau}\right]\right\}. \tag{8.70}$$

(This equation is to be compared with Eq. (6.75b).) Substituting Eqs. (8.60), (8.66), and (8.70) into Eq. (8.2) gives the supereikonal expression for the pressure [80,81]:

$$p(\mathbf{x}) = \frac{1}{4\pi}\left[\frac{k_o}{2\pi i}\right]^{1/2} \int_0^\infty \frac{d\tau}{\tau^{3/2}} \exp\left\{\frac{ik_o}{2}\left[\tau + \frac{|\mathbf{x} - \mathbf{x}_s|^2}{\tau} - 2\tau I(\tau) + i\epsilon\tau\right]\right\}. \tag{8.71}$$

where

$$I(\tau) \equiv \int \frac{d\mathbf{k}}{(2\pi)^3} U(\mathbf{k}) \int_0^1 d\beta \exp\left\{i\mathbf{k} \cdot [\mathbf{x}_s + \beta(\mathbf{x} - \mathbf{x}_s)] - \frac{i\mathbf{k}^2}{2k_o} \tau\beta(1-\beta)\right\}. \quad (8.72)$$

The Rytov approximation for Φ led to the supereikonal approximation for $p(\mathbf{x})$. On the other hand, the Rytov approximation applied directly to $p(\mathbf{x})$ gives

$$p(\mathbf{x}) = G_o(\mathbf{x} - \mathbf{x}_s)\exp\left[\frac{-2k_o^2}{G_o(\mathbf{x} - \mathbf{x}_s)} \int d\mathbf{x}'\, G(\mathbf{x} - \mathbf{x}')U(\mathbf{x}')G(\mathbf{x}' - \mathbf{x}_s)\right], \quad (8.73)$$

with

$$G_o(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|} e^{ik_o|\mathbf{x}|}. \quad (8.74)$$

(The exponent in Eq. (8.73) is the Born term divided by G_o.) We now ask: What is the relationship between the Rytov approximation for $p(\mathbf{x})$ and the supereikonal approximation? To answer this question, we return to Eq. (8.68) for $Q^{(1)}$ and observe that the integral over τ' may be evaluated with the aid of the elementary formula

$$\int_0^\tau \frac{d\tau'}{[\tau'(\tau - \tau')]^{3/2}} \exp\left[\frac{ik_o}{2}\left(\frac{\mathbf{A}^2}{\tau - \tau'} + \frac{\mathbf{B}^2}{\tau'}\right)\right]$$
$$= \left(\frac{2\pi i}{k_o\tau}\right)^{1/2} \left(\frac{|\mathbf{A}| + |\mathbf{B}|}{\tau|\mathbf{A}||\mathbf{B}|}\right) \exp\left[\frac{ik_o}{2\tau}(|\mathbf{A}| + |\mathbf{B}|)^2\right]. \quad (8.75)$$

The result is

$$Q^{(1)} = -\frac{2k_o^2}{4\pi} \int d\mathbf{x}'\, U(\mathbf{x}') \left(\frac{|\mathbf{x} - \mathbf{x}'| + |\mathbf{x}' - \mathbf{x}_s|}{|\mathbf{x} - \mathbf{x}'||\mathbf{x}' - \mathbf{x}_s|}\right)$$
$$\times \exp\left\{\frac{ik_o}{2\tau}[(|\mathbf{x} - \mathbf{x}'| + |\mathbf{x}' - \mathbf{x}_s|)^2 - |\mathbf{x} - \mathbf{x}_s|^2]\right\}. \quad (8.76)$$

In the straight-line geometric-optics approximation the integration over $\mathbf{x}'$ is restricted to the straight line joining the points $\mathbf{x}_s$ and $\mathbf{x}$, giving $|\mathbf{x} - \mathbf{x}'| + |\mathbf{x}' - \mathbf{x}_s| = |\mathbf{x} - \mathbf{x}_s|$. Here we shall assume that large-angle scattering is unimportant, so that the difference $|\mathbf{x} - \mathbf{x}'| + |\mathbf{x}' - \mathbf{x}_s| - |\mathbf{x} - \mathbf{x}_s|$ is small compared to $|\mathbf{x} - \mathbf{x}_s|$. With this assumption Eq. (8.76) becomes

$$Q^{(1)} = -\frac{2k_o^2}{4\pi} \int d\mathbf{x}\, U(\mathbf{x}') \left(\frac{|\mathbf{x} - \mathbf{x}_s|}{|\mathbf{x} - \mathbf{x}'||\mathbf{x}' - \mathbf{x}_s|}\right)$$
$$\times \exp\left\{\frac{ik_o|\mathbf{x} - \mathbf{x}_s|}{\tau}(|\mathbf{x} - \mathbf{x}'| + |\mathbf{x} - \mathbf{x}_s| - |\mathbf{x} - \mathbf{x}_s|)\right\}. \quad (8.77)$$

Let us return to the expression for $p(\mathbf{x})$ in the supereikonal approximation:

$$p(\mathbf{x}) = \frac{1}{4\pi}\left(\frac{k_o}{2\pi i}\right)^{1/2} \int_0^\infty \frac{d\tau}{\tau^{3/2}} \exp\left[\frac{ik_o}{2}\left(\tau + \frac{|\mathbf{x} - \mathbf{x}_s|^2}{\tau}\right) + Q^{(1)}\right]. \quad (8.78)$$

The integral over τ is now evaluated by the method of stationary-phase (Appendix B). In determining the stationary phase point, the dependence of $Q^{(1)}$ on τ is ignored. We find

$$p(\mathbf{x}) = \frac{e^{ik_o|\mathbf{x}-\mathbf{x}_s|}}{4\pi|\mathbf{x}-\mathbf{x}_s|} \exp Q^{(1)}|_{\tau=|\mathbf{x}-\mathbf{x}_s|}. \tag{8.79}$$

By using Eq. (8.77) for $Q^{(1)}$, we readily obtain the Rytov approximation, Eq. (8.73). One may also obtain Eq. (8.73) directly from Eqs. (8.76) and (8.78) by using a saddle-point approximation to evalute the integral over τ.

We conclude this section by noting the straight-line geometric-optics approximation for $p(\mathbf{x})$ is easily obtained from the supereikonal approximation. We drop the $\mathbf{k}^2$ term in Eq. (8.72) for $I(\tau)$ and find

$$I = \int_0^1 d\beta\, U(\mathbf{x}_s + \beta(\mathbf{x}-\mathbf{x}_s)) \tag{8.80}$$

independent of τ. Substituting this into Eq. (8.71) and evaluating the integral over τ by the method of stationary phase, we get

$$p(\mathbf{x}) = G_o(\mathbf{x}-\mathbf{x}_s) \exp\left[-ik_o \int_0^{|\mathbf{x}-\mathbf{x}_s|} dt\, U\left(x_s + t\frac{(\mathbf{x}-\mathbf{x}_s)}{|\mathbf{x}-\mathbf{x}_s|}\right)\right]. \tag{8.81}$$

In obtaining Eq. (8.81), we have considered only the term

$$\frac{ik_o}{2}\left(\tau + \frac{1}{\tau}\right)$$

in determining the stationary-phase point. (Of course in this case the integral over τ can be evaluated exactly.)

9. THE PARABOLIC APPROXIMATION

9.1 Preliminary Discussion

We have derived path-integral representations for the Helmholtz equation and for the two- and three-dimensional parabolic equations. In this section we will use these representations to discuss the parabolic approximation.

To see how one might develop the parabolic approximation, let us consider the expression for $p(\mathbf{x})$ in the straight-line geometric-optics approximation, Eq. (8.81):

$$p(\mathbf{x}) = G_o(\mathbf{x}-\mathbf{x}_s) \exp\left[-ik_o \int_0^{|\mathbf{x}-\mathbf{x}_s|} dr\, U\left(\mathbf{x}_s + r\frac{(\mathbf{x}-\mathbf{x}_s)}{|\mathbf{x}-\mathbf{x}_s|}\right)\right]. \tag{9.1}$$

We derived this expression using a two-step process: We made a straight-line geometric-optics approximation to Φ of Eq. (8.2); then we evaluated the integral over τ by using the method of stationary phase, determining the stationary-phase point from that part of the phase in Eq. (8.2) which does not depend on U.

We now write Eq. (9.1) in the form

$$p(\mathbf{x}) = G_o(\mathbf{x} - \mathbf{x}_s)\psi(|\mathbf{x} - \mathbf{x}_s|), \tag{9.2}$$

where

$$\psi(r) = \exp\left[-ik_o \int_0^r dr' U\left(\mathbf{x}_s + r' \frac{(\mathbf{x} - \mathbf{x}_s)}{|\mathbf{x} - \mathbf{x}_s|}\right)\right]. \tag{9.3}$$

Now $\psi(r)$ satisfies the differential equation

$$-2ik_o\partial_r\psi(r) = \{k_o^2[n^2(r) - 1]\}\psi(r), \tag{9.4}$$

where

$$n(r) = c_o/c\left(\mathbf{x}_s + \frac{r}{R_3}(\mathbf{x} - \mathbf{x}_s)\right), \tag{9.5a}$$

in which

$$R_3 \equiv \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2}. \tag{9.5b}$$

We compare these equations with the two-dimensional parabolic equation (Eq. (3.5))

$$-2ik_o\partial_r\psi(r, z) = \{\partial_z^2 + k_o^2[n^2(r, z) - 1]\}\psi(r, z), \tag{9.6}$$

where

$$n(r, z) = c_o/c\left[x_s + \frac{r}{R_2}(x - x_s), y_s + \frac{r}{R_2}(y - y_s), z\right], \tag{9.7a}$$

in which

$$R_2 \equiv \sqrt{(x - x_s)^2 + (y - y_s)^2}, \tag{9.7b}$$

and with the parabolic equation in three dimensions (Eq. (3.12a))

$$-2ik_o\partial_r\psi(r, y, z) = \{\partial_y^2 + \partial_z^2 + k_o^2[n^2(r, y, z) - 1]\}\psi(r, y, z), \tag{9.8}$$

where

$$n(r, y, z) = c_o/c\left(x_s + \frac{r}{R_1}(x - x_s), y, z\right), \tag{9.9a}$$

in which

$$R_1 \equiv \sqrt{(x - x_s)^2}. \tag{9.9b}$$

The similarity between these three sets of equations is obvious and suggests the following scheme:

- To derive a "parabolic equation" with *no* transverse second derivatives, we calculate Φ using straight-line geometric optics and evaluate the integral over τ using the method of stationary phase.

- To derive a parabolic equation with *one* transverse second derivative, we calculate Φ using a straight-line geometric-optics approximation applied to *two* coordinates and then evaluate the integral over τ as before.

• To obtain a parabolic equation with *two* transverse second derivatives, we apply straight-line geometric optics to *one* coordinate and then make the stationary-phase approximation.

In the next subsection we will demonstrate the correctness of this scheme. We will use the first path-integral representation, Eq. (8.15), of the previous section. Along the way we will discuss the concept of additivity of the action.

A standard reference on the parabolic approximation is the paper by Klyatskin and Tartarskii [48]. In subsection 9.3 we will review their calculation and compare it with the calculation in subsection 9.2. Klyatskin and Tatarskii were interested in the application of the parabolic equation to problems in the theory of propagation through a random medium. To discuss this work, we must consider a model for the random fluctuations on the sound speed. The first part of subsection 9.3 will be devoted to this model and to an approximation intimately related to the parabolic approximation called the Markov approximation [76, Ch. 5]. In subsection 9.4 we will derive an improved two-dimensional parabolic equation by relaxing the geometric-optics approximation. In the final subsection we will discuss corrections to the stationary-phase approximation.

Before proceeding, we list the basic equations of the path-integral representation we will use:

$$p(\mathbf{x}) = \frac{i}{2k_o} \int_0^\infty d\tau \exp\left(\frac{ik_o\tau}{2}\right) \Phi(\tau, \mathbf{x}|0, \mathbf{x}_s), \tag{9.10}$$

and

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \int D[\mathbf{x}(\sigma)] \exp\left\{ik_o \int_0^\tau d\sigma \left[\frac{1}{2}\left(\frac{d\mathbf{x}}{d\sigma}\right)^2 - U(\mathbf{x}(\sigma))\right]\right\}, \tag{9.11}$$

where $\mathbf{x}(0) = \mathbf{x}_s$ and $\mathbf{x}(\tau) = \mathbf{x}$. The path integral is normalized so that

$$\int D[\mathbf{x}(\sigma)] \exp\left[\frac{ik_o}{2} \int_0^\tau d\sigma \left(\frac{d\mathbf{x}}{d\sigma}\right)^2\right] = \left(\frac{k_o}{2\pi i\tau}\right)^{3/2} \exp\left[\frac{ik_o\tau}{2}\left(\frac{\mathbf{x} - \mathbf{x}_s}{\tau}\right)^2\right]. \tag{9.12}$$

Furthermore Φ satisfies

$$-2ik_o\partial_\tau \Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \{\nabla^2 - 2k_o^2 U(\mathbf{x})\}\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) \tag{9.13}$$

with the initial condition

$$\Phi(0, \mathbf{x}|0, \mathbf{x}_s) = \delta^{(3)}(\mathbf{x} - \mathbf{x}_s). \tag{9.14}$$

The function U is defined, as usual, by

$$U(\mathbf{x}) = -\frac{1}{2}[n^2(\mathbf{x}) - 1]. \tag{9.15}$$

9.2 Straight-Line Geometric Optics and the Method of Stationary Phase

We will consider first the derivation of the two-dimensional parabolic equation. According to the scheme outlined in the previous subsection this requires applying straight-line geometric optics to the horizontal coordinates. Referring to Eq. (9.11), we set

$$x_\perp(\sigma) = s_\perp + \frac{\sigma}{\tau}(x_\perp - s_\perp) + x'_\perp(\sigma)$$

and

$$z(\sigma) = z'(\sigma). \tag{9.16}$$

Here we are using a $\perp$ subscript to indicate a two-dimensional vector in the horizontal plane:

$$x_\perp(\sigma) = (x(\sigma), y(\sigma)),$$
$$s_\perp = (x_s, y_s),$$
$$R = \sqrt{(x - x_s)^2 + (y - y_s)^2} = |x_\perp - s_\perp|, \tag{9.17}$$

etc. If we further define a two-dimensional unit vector in the horizontal direction of propagation

$$e_\perp = \frac{1}{R}(x - x_s, y - y_s) = (x_\perp - s_\perp)/R, \tag{9.18}$$

we have

$$x_\perp(\sigma) = s_\perp + \frac{R\sigma}{\tau} e_\perp + x'_\perp(\sigma). \tag{9.19}$$

Equation (9.11) now reads

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \exp\left[\frac{ik_o R^2}{2\tau}\right] \int D[\mathbf{x}'(\sigma)] \exp\left[\frac{ik_o}{2}\int_o^\tau d\sigma \left(\frac{d\mathbf{x}'}{d\sigma}\right)^2\right]$$

$$\times \exp\left[-ik_o \int_0^\tau d\sigma\, U\left(s_\perp + \frac{\sigma R}{\tau} e_\perp + x'_\perp(\sigma), z'(\sigma)\right)\right]. \tag{9.20}$$

Horizontal straight-line geometric optics follows from ignoring the dependence of U on $x'_\perp(\sigma)$. With this approximation the integral over $x'_\perp(\sigma)$ is trivial, and we obtain

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \left(\frac{k_o}{2\pi i\tau}\right) \exp\left(\frac{ik_o R^2}{2\tau}\right) \Gamma(\tau, z|0, z_s), \tag{9.21}$$

where

$$\Gamma(\tau, z|0, z_s) = \int D[z(\sigma)] \exp\left\{ik_o \int_0^\tau d\sigma \left[\frac{1}{2}\left(\frac{dz}{d\sigma}\right)^2 - U\left(s_\perp + \frac{\sigma R}{\tau} e_\perp, z(\sigma)\right)\right]\right\}. \tag{9.22}$$

The expression for the pressure is

$$p(\mathbf{x}) = \frac{1}{4\pi}\int_0^\infty \frac{d\tau}{\tau} \exp\left[\frac{ik_o}{2}\left(\tau + \frac{R^2}{\tau}\right)\right] \Gamma(\tau, z|0, z_s). \tag{9.23}$$

We now can learn something new about path integrals. One might expect us to use Eq. (9.22) to derive a parabolic equation for $\Gamma(\tau, z|0, z_s)$. In fact the path integral in Eq. (9.22) does not in general satisfy a simple parabolic equation, and the reason is due to an important concept called the *additivity of the action.*

To illustrate this concept, consider the form of the path integral which solves the parabolic equation

$$\Psi(r, z | r', z') = \int D[z(s)] \exp[ik_0 A(z(s))], \tag{9.24}$$

where

$$A(z(s)) = \int_{r'}^{r} ds\, L(s, z(s)). \tag{9.25}$$

The most important characteristic of this path integral is that the action A is additive:

$$\begin{aligned} A(z(s)) &= \int_{r'}^{r''} ds\, L(s, z_1(s)) + \int_{r''}^{r} ds\, L(s, z_2(s)) \\ &= A_1(z_1(s)) + A_2(z_2(s)), \end{aligned} \tag{9.26}$$

where $z_1(s) = z(s)$ for $r' \leqslant s \leqslant r''$ and $z_2(s) = z(s)$ for $r'' \leqslant s \leqslant r$. The component actions A_1 and A_2 are identical in form to A. They differ only in the limits of integration and in the endpoint conditions satisfied by paths. It is this additivity which leads to the composition law, the parabolic equation, and the marching algorithms for numerically evaluating the path integral. If the action is not additive, one has none of these things. Generally speaking, path integrals with nonadditive actions are very difficult to handle. The application of Feynman's theory to the problem of propagation through a random medium has not been particularly fruitful, until recently, because one always encountered path integrals with nonadditive actions [82]. One of Dashen's most important contributions [5] is that he showed these nonadditive actions become additive if one makes the Markov approximation, an approximation which, as we shall see, is often no stronger than the parabolic approximation. With additive actions Dashen was able to develop a rather complete theory of propagation through random media, one which seems to be in excellent agreement with experiment [83].

To place this discussion in context, we observe that the path integral of Eq. (9.22) has the form

$$\int D[(z(\sigma)] \exp\left[ik_0 \int_0^{\tau} d\sigma\, L(\sigma, z(\sigma), \tau)\right]. \tag{9.27}$$

The action is not additive, because τ is present in L. If the sound speed is a function of just depth, L is independent of τ, the action is additive, and one easily obtains

$$-2ik_0\partial_\tau \Gamma(\tau, z|0, z_s) = \{\partial_z^2 - 2k_0^2 U(z)\}\Gamma(\tau, z|0, z_s), \tag{9.28}$$

with

$$\Gamma(0, z|0, z_s) = \delta(z - z_s). \tag{9.29}$$

To find the differential equation satisfied by Γ in the general case, we return to the discrete form of the path integral

$$\begin{aligned} \Gamma(\tau, z|0, z_s) \simeq \prod_{i=1}^{N} \left[\frac{k_0}{2\pi i(\sigma_i - \sigma_{i-1})}\right]^{N/2} \int dz_1 \ldots dz_{N-1} \exp\Bigg\{ik_0 \\ \times \sum_{i=1}^{N}\left[\frac{1}{2}\frac{(z_i - z_{i-1})^2}{(\sigma_i - \sigma_{i-1})} - (\sigma_i - \sigma_{i-1})U\left(s_\perp + \frac{\sigma_i R}{\tau} e_\perp, z_i\right)\right]\Bigg\}. \end{aligned} \tag{9.30}$$

where $z_N = z$ and $\sigma_N = \tau$. A straightforward calculation gives

$$[-2ik_o\partial_\tau - \partial_z^2 + 2k_o^2 U(\mathbf{x})]\Gamma(\tau, z|0, z_s)$$

$$= \frac{2k_o^2 R}{\tau^2} \int D[z(\sigma)] \exp\left\{ik_o \int_0^\tau d\sigma \left[\frac{1}{2}\left(\frac{dz}{d\sigma}\right)^2 - U\left(s_\perp + \frac{\sigma R}{\tau} e_\perp, z(\sigma)\right)\right]\right\}$$

$$\times \int_0^\tau d\sigma' \sigma' e_\perp \cdot \nabla_\perp U\left(s_\perp + \frac{\sigma R}{\tau} e_\perp, z(\sigma)\right) \tag{9.31}$$

where $\nabla_\perp = (\partial_x, \partial_y)$. If U is independent of $x_\perp$, then $\nabla_\perp U = 0$ and Eq. (9.31) reduces to Eq. (9.28).

The best way to find Γ is not by solving Eq. (9.31) but by introducing a more general path integral:

$$\Gamma_\tau'(t, z|0, z_s) = \int D[z(\sigma)] \exp\left\{ik_o \int_0^t d\sigma \left[\frac{1}{2}\left(\frac{dz}{d\sigma}\right)^2 - U\left(s_\perp + \frac{\sigma R}{\tau} e_\perp, z(\sigma)\right)\right]\right\}. \tag{9.32}$$

This path integral is more complicated, since it has an additional dependence on the variable t. However

$$\Gamma_\tau'(\tau, z|0, z_s) = \Gamma(\tau, z|0, z_s), \tag{9.33}$$

and, since the action in Eq. (9.32) is additive, it is not difficult to show Γ' obeys a simple parabolic equation:

$$-2ik_o\partial_t\Gamma_\tau'(t, z|0, z_s) = \left[\partial_z^2 - 2k_o^2 U\left(s_\perp + \frac{tR}{\tau} e_\perp, z\right)\right]\Gamma_\tau'(t, z|0, z_s), \tag{9.34a}$$

with

$$\Gamma_\tau'(0, z|0, z_s) = \delta(z - z_s). \tag{9.34b}$$

Unlike differential equations (9.28) and (9.31), when Eqs. (9.34) are being solved τ is treated as a parameter. We prefer a slightly different form of Eqs. (9.34) obtained by letting

$$t = \frac{\tau}{R} r \tag{9.35a}$$

and

$$\Psi_\tau(r, z|0, z_s) = \Gamma_\tau'\left(\frac{\tau r}{R}, z|0, z_s\right). \tag{9.35b}$$

Equations (9.23), (9.34) now are

$$p(\mathbf{x}) = \frac{1}{4\pi} \int_0^\infty \frac{d\tau}{\tau} \exp\left[\frac{ik_o}{2}\left(\tau + \frac{R^2}{\tau}\right)\right] \Psi_\tau(R, z|0, z_s) \tag{9.36a}$$

where

$$\frac{-2ik_o}{(\tau/R)} \partial_r\Psi_\tau(r, z|0, z_s) = \{\partial_z^2 + k_o^2[n^2(r, z) - 1]\}\Psi_\tau(r, z|0, z_s), \tag{9.36b}$$

with $n(r, z) = n(s_\perp + re_\perp, z)$ and

$$\Psi_\tau(0, z|0, z_s) = \delta(z - z_s). \tag{9.36c}$$

The following points are worth discussing.

- Equations (9.36) were first derived in Ref. 84. In that paper Feynman path integrals were not used. Instead the pressure field $p(\mathbf{x})$ was expanded in a power series in U using standard perturbation theory. Each term in the series was then approximated by applying the straight-line geometric-optics approximation (the eikonal approximation) to the horizontal coordinates, and the series was resummed. The calculation made no assumptions about the depth variation of $p(\mathbf{x})$ or of the sound speed. Hence Eqs. (9.36) are valid regardless of the depth-dependent boundary conditions imposed on the pressure.

- In the equation satisfied by Ψ_τ the variable of integration τ enters as a parameter; the equation is solved as if it were constant. If $\tau = R$, Eqs. (9.36b) and (9.36c) reduce to the propagator defined by Eqs. (3.9) rather than the field ψ defined by Eq. (3.8). The point is that the depth-dependent initial field $h(z)$ plays no role when one considers a point source, that is, when the Helmholtz equation has a spatial δ-function on the right-hand side.

- In deriving these equations, we have assumed only that straight-line geometric optics may be applied to the horizontal coordinates. If the sound speed does not vary with the horizontal coordinates, this type of approximation is exact. Therefore Eqs. (9.36) should be exact if the sound speed is a function simply of depth. this is easily demonstrated. If $c(\mathbf{x}) = c(z)$, then

$$p(\mathbf{x}) = \frac{i}{4} \sum_n Z_n(z) Z_n(z_s) H_o^{(1)}(k_n R), \tag{9.37}$$

where

$$\left[\frac{d^2}{dz^2} + k_o^2 n^2(z)\right] Z_n(z) = k_n^2 Z_n(z). \tag{9.38}$$

These modal functions (which we assume possess only a discrete spectrum) are orthonormal,

$$\int_0 dz\, Z_n(z) Z_m(z) = \delta_{nm}, \tag{9.39}$$

and complete,

$$\sum_n Z_n(z) Z_n(z') = \delta(z - z'). \tag{9.40}$$

From Eq. (A12) we have then

$$p(\mathbf{x}) = \frac{1}{4\pi} \int_0^\infty \frac{d\lambda}{\lambda} \sum_n Z_n(z) Z_n(z_s) \exp\left[\frac{ik_n R}{2}\left[\lambda + \frac{1}{\lambda}\right]\right]. \tag{9.41}$$

Letting $\lambda = k_n \tau / R k_o$ gives Eq. (9.36a) with

$$\Psi_\tau(r, z|0, z_s) = \sum_n Z_n(z) Z_n(z_s) \exp\left[\frac{i\tau r}{2k_o R}(k_n^2 - k_o^2)\right]. \tag{9.42}$$

Using Eqs. (9.38) and (9.40), it is easy to see Eqs. (9.36b) and (9.36c) are satisfied.

• We have emphasized the difficulty of writing a differential equation for $\Gamma(\tau, z|0, z_s)$ when the sound speed depends on x and y. However for a range-independent sound speed we immediately obtained Eqs. (9.28) and (9.29). Hence for this case Eqs. (9.28), and (9.29) must have a relationship to Eqs. (9.36b) and (9.36c). To exhibit this relationship, we note that, if $n^2(s_\perp + re_\perp, z) = n^2(z)$, the form of Eq. (9.36b) dictates Ψ_τ can depend on r only through the product $r\tau$ (Eq. (9.42)):

$$\Psi_\tau(r, z|0, z_s) = \text{a function of } r\tau.$$

We therefore have the symmetry relation

$$r\partial_r \Psi_\tau(r, z|0, z_s) = \tau\partial_\tau \Psi_\tau(r, z|0, z_s). \tag{9.43}$$

Substituting this into Eq. (9.36b), we get

$$\frac{-2ik_o}{(r/R)}\partial_\tau \Psi_\tau(r, z|0, z_s) = \{\partial_z^2 + k_o^2[n^2(z) - 1]\}\Psi_\tau(r, z|0, z_s). \tag{9.44}$$

We now let $r = R$:

$$-2ik_o\partial_\tau \Psi_\tau(R, z|0, z_s) = \{\partial_z^2 + k_o^2[n^2(z) - 1]\}\Psi_\tau(R, z|0, z_s). \tag{9.45}$$

Because $\Psi_\tau(R, z|0, z_s) = \Gamma(\tau, z|0, z_s)$, Eq. (9.45) is equivalent to Eq. (9.28). In the general case, Ψ_τ will not be simply a function of $r\tau$ because of the additional dependence of the index of refraction on r. Therefore

$$(r\partial_r - \tau\partial_\tau)\Psi_\tau(r, z|0, z_s)$$

$$= \text{some function due to the range dependence of the sound speed;} \tag{9.46}$$

that is, the symmetry is broken by the range-dependent part of the sound speed. Since the right-hand side of Eq. (9.46) is nonzero (and complicated), it is not in general possible to write $p(\mathbf{x})$ as an integral over τ of an exponential times an object which obeys a simple parabolic equation with a first derivative in τ. (In subsection 9.5 we will use Eq. (9.46) as the starting point of a calculation of the corrections to the stationary-phase approximation due to the range-dependent part of the sound speed.)

To complete the analysis, the integral over τ in Eq. (9.36a) is approximated by the method of stationary phase (Appendix B). This approximation requires that $k_oR \gg 1$ and that Ψ_τ be a smoothly varying function of τ in the region of the stationary-phase point $\tau = R$. We easily obtain the parabolic approximation

$$p(\mathbf{x}) = \frac{1}{4\pi}\left(\frac{2\pi i}{k_oR}\right)^{1/2} e^{ik_oR}\Psi(R, z|0, z_s), \tag{9.47}$$

where $\Psi = \Psi_{\tau=R}$ satisfies the parabolic equation given by Eq. (3.9a) with the δ-function initial condition as given by Eq. (3.9b).

We now consider the derivation of the three-dimensional parabolic equation. Application of the straight-line geometric-optics approximation to only the x coordinate in Eq. (9.11) yields

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \left(\frac{k_o}{2\pi i\tau}\right)^{1/2} \exp\left[\frac{ik_o(x - x_s)^2}{2\tau}\right] \Gamma_\tau'(\tau, \hat{\rho}|0, \hat{\rho}_s), \tag{9.48}$$

where

$$\Gamma_\tau'(t,\hat{\rho}|0,\hat{\rho}_s) = \int D[\hat{\rho}(\sigma)]\exp\left\{ik_o\int_0^t d\sigma\left[\frac{1}{2}\left(\frac{d\hat{\rho}}{d\sigma}\right)^2 - U\left(x_s + \frac{\sigma}{\tau}(x - x_s), \hat{\rho}(\sigma)\right)\right]\right\}. \quad (9.49)$$

Here $\hat{\rho}(\sigma) = (y(\sigma), z(\sigma))$, so $\hat{\rho}(t) = \hat{\rho} = (y, z)$ and $\hat{\rho}(o) = \hat{\rho}_s = (y_s, z_s)$. Now Γ' satisfies the equations

$$-2ik_o\partial_t\Gamma_\tau'(t,\hat{\rho}|0,\hat{\rho}_s) = \left[\hat{\nabla}^2 - 2k_o^2 U\left(x_s + \frac{t}{\tau}(x - x_s), \hat{\rho}\right)\right]\Gamma_\tau'(t,\hat{\rho}|0,\hat{\rho}_s), \quad (9.50a)$$

and

$$\Gamma_\tau'(0,\hat{\rho}|0,\hat{\rho}_s) = \delta^{(2)}(\hat{\rho} - \hat{\rho}_s). \quad (9.50b)$$

Defining

$$t = \frac{\tau}{(x - x_s)} r \quad (9.51a)$$

$$\Psi_\tau(r,\hat{\rho}|0,\hat{\rho}_s) = \Gamma_\tau'\left(\frac{\tau r}{(x - x_s)}, \hat{\rho}|0,\hat{\rho}_s\right), \quad (9.51b)$$

we have the system of equations

$$p(\mathbf{x}) = \frac{1}{4\pi}\left(\frac{2\pi i}{k_o}\right)^{1/2}\int_0^\infty \frac{d\tau}{\tau^{1/2}}\exp\left\{\frac{ik_o}{2}\left[\tau + \frac{(x - x_s)^2}{\tau}\right]\right\}\Psi_\tau(x - x_s,\hat{\rho}|0,\hat{\rho}_s), \quad (9.52a)$$

$$\frac{-2ik_o}{[\tau/(x - x_s)]}\partial_r\Psi_\tau(r,\hat{\rho}|0,\hat{\rho}_s) = [\hat{\nabla}^2 - 2k_o^2 U(x_s + r,\hat{\rho})]\Psi_\tau(r,\hat{\rho}|0,\hat{\rho}_s), \quad (9.52b)$$

and

$$\Psi_\tau(0,\hat{\rho}|0,\hat{\rho}_s) = \delta^{(2)}(\hat{\rho} - \hat{\rho}_s). \quad (9.52c)$$

The stationary-phase approximation is straight-forward and yields

$$p(\mathbf{x}) = \frac{i}{2k_o}e^{ik_o(x - x_s)}\Psi(x - x_s,\hat{\rho}|0,\hat{\rho}_s), \quad (9.53)$$

where $\Psi = \Psi_{\tau = x - x_s}$ satisfies Eqs. (3.12).

As one final topic we consider the significance of the parabolic approximation for propagation in an infinite, homogeneous ocean where $U = 0$. We have

$$\Psi(R, z|0, z_s) = \left(\frac{k_o}{2\pi iR}\right)^{1/2}\exp\left[\frac{ik_oR}{2}\left(\frac{z - z_s}{R}\right)^2\right]. \quad (9.54)$$

Substituting this into Eq. (9.47) we get

$$p(\mathbf{x}) = \frac{1}{4\pi R}\exp\left\{ik_o\left[R + \frac{(z - z_s)^2}{2R}\right]\right\}. \quad (9.55)$$

This equation is to be compared with the exact result

$$p(\mathbf{x}) = \frac{1}{4\pi\sqrt{R^2 + (z - z_s)^2}}\exp\left[ik_o\sqrt{R^2 + (z - z_s)^2}\right] \quad (9.56)$$

Obviously the parabolic approximation requires

$$\left(\frac{z - z_s}{R}\right)^2 \ll 1 \tag{9.57a}$$

and

$$\frac{k_0 R}{8}\left(\frac{z - z_s}{R}\right)^4 \ll 2\pi. \tag{9.57b}$$

The results for the three-dimensional equation are similar.

9.3 The Calculation of Klyatskin and Tatarskii: Random Fluctuations in the Sound Speed

We consider in this subsection the situation in which temperature fluctuations produce a random component in the sound speed. The function U is then a random variable which may be written in the form

$$U(\mathbf{x}) = \langle U(\mathbf{x}) \rangle + (U(\mathbf{x}) - \langle U(\mathbf{x}) \rangle), \tag{9.58}$$

where $\langle \ldots \rangle$ represents an average over a large number of measurements of U. This decomposition defines deterministic and random components

$$U_d(\mathbf{x}) = \langle U(\mathbf{x}) \rangle \tag{9.59a}$$

and

$$U_r(\mathbf{x}) = U(\mathbf{x}) - \langle U(\mathbf{x}) \rangle . \tag{9.59b}$$

We shall use a particularly simple model for U_d and U_r. The ocean is far too complicated for this model to be even approximately valid, and calculations based on it can at best give only the correct orders of magnitude. We consider it not because we fell it will lead to a description of acoustic fluctuations but simply so we may connect our analysis of the parabolic approximation with a previous study.

We assume:

- The deterministic component is a function of all three spatial coordinates:

$$U_d = U_d(x, y, z). \tag{9.60}$$

- The random component is a Gaussian random variable. Hence, all moments of U_r can be expressed in terms of the correlation function

$$B(\mathbf{x}; \mathbf{x}') = \langle U_r(\mathbf{x}) U_r(\mathbf{x}') \rangle . \tag{9.61}$$

Of course U_r has a zero mean:

$$\langle U_r(\mathbf{x}) \rangle = 0. \tag{9.62}$$

- The temperature fluctuations are homogeneous but not necessarily isotropic:

$$B(\mathbf{x}; \mathbf{x}') = B(|x - x'|, |y - y'|, |z - z'|). \tag{9.63}$$

- The fluctuations along the coordinate directions are characterized by scale paramenters L_x, L_y, L_z. That is, for $|\mathbf{x}|$ small,

$$B(\mathbf{x};\,0) = \sigma_U^2\left[1 - \frac{1}{2}\left(\frac{x}{L_x}\right)^2 - \frac{1}{2}\left(\frac{y}{L_y}\right)^2 - \frac{1}{2}\left(\frac{z}{L_z}\right)^2 + \ldots\right], \tag{9.64}$$

where

$$\sigma_U^2 = B(0;\,0) = <(U_r(\mathbf{x})^2)>. \tag{9.65}$$

• the correlation function possesses separability along the direction of propagation. That is, if propagation is along the x axis,

$$B(|x|,|y|,|z|) = B(|x|,0,0,)B(0,|y|,|z|). \tag{9.66}$$

It will be convenient to rewrite this equation as

$$B(|x|,|y|,|z|) = \zeta(|x|)\tilde{B}(|y|,|z|) \tag{9.67}$$

where

$$\zeta(|x|) = B(|x|,0,0) \;/\; \int_{-\infty}^{+\infty} dx\, B(|x|,0,0) \tag{9.68a}$$

and

$$\tilde{B}(|y|,|z|) = B(o,|y|,|z|) \int_{-\infty}^{\infty} dx\, B(|x|,0,0) \tag{9.68b}$$

• From dimensional considerations we have

$$\tilde{B}(0,0) \simeq L_x \sigma_U^2 \tag{9.69}$$

and

$$\int_{-\infty}^{\infty} dx\, x^{2n}\zeta(|x|) \simeq L_x^{2n} \tag{9.70}$$

This last relation will be used only for n small, ($n = 1$ or 2).

When one has a random sound speed, the parabolic approximation is intimately related to another approximation called the Markov approximation [76, Ch. 5]. The Markov approximation states that in the calculation of the average of any physical quantity the function ζ of Eq. (9.67) may be replaced by a δ function

$$\zeta(|x - x'|) \rightarrow \delta(x - x'). \tag{9.71}$$

Before discussing conditions under which the Markov approximation is valid, let us consider what it means. If the correlation function has a Gaussian shape, then

$$\zeta(|x|) = \frac{1}{\sqrt{2\pi}L_x}\exp\left[-\frac{1}{2}\left(\frac{x}{L_x}\right)^2\right]. \tag{9.72}$$

This expression approaches a δ function as $L_x \rightarrow 0$. Consequently the Markov approximation in a problem is a statement about the size of L_x in relation to the other dimensions which characterize the effect of fluctuations in the pressure due to fluctuations in the sound speed *along the direction of propagation.* If L_x is small compared to these dimensions, it may be set equal to zero, giving Eq. (9.71).

In the theory of propagation through a random medium, the conditions under which some approximation is valid are determined by comparing expressions for the average value of

products of the pressure field. That is, expressions for $<p>$, $<pp^*>$, $<pp>$, $<ppp^*>$, ... before and after making the approximation are compared. In principle this comparison should be done for all average values of experimental interest. In practice it is usually done only for the average value of the field $<p>$ and for the mutual coherence function $<pp^*>$. In this report we shall consider only $<p>$. Consequently we will not be able to derive all the conditions associated with a given approximation.

The standard technique [76, Ch. 5] for determining the conditions for the validity of the Markov approximation is to assume first that the three-dimensional parabolic approximation is valid. We therefore return to the path integral which solves the three-dimensional parabolic equation

$$\Psi(X, \hat{\rho}|0, \hat{\rho}_s) = \int D[\hat{\rho}(r)] \exp\left\{ik_o \int_0^X dr\left[\frac{1}{2}\left(\frac{d\hat{\rho}}{dr}\right)^2 - U(r, \hat{\rho}(r))\right]\right\}. \tag{9.73}$$

Here we assume the direction of propagation is along the positive x axis, $X = x - x_s$ is the propagation distance, $\hat{\rho} = (y, z)$, and $\hat{\rho}_s = (y_s, z_s)$.

We shall calculate $<\Psi>$ by assuming the Markov approximation and then by assuming ζ is a sharply peaked function of its argument rather than a δ function. Comparison of the two expressions will give the first-order correction to the Markov approximation and hence a condition for its validity.

Taking the average of Eq. (9.73), we obtain

$$<\Psi> = \int D[\hat{\rho}(r)] \exp\left\{ik_o \int_0^X dr\left[\frac{1}{2}\left(\frac{d\hat{\rho}}{dr}\right)^2 - U_d(r, \hat{\rho}(r))\right]\right\} \epsilon(\hat{\rho}(r)), \tag{9.74}$$

where

$$\epsilon(\hat{\rho}(r)) = <\exp\left[-ik_o \int_0^X dr\, U_r(r, \hat{\rho}(r))\right]>. \tag{9.75}$$

Since U_r is a zero-mean Gaussian random variable, we can write

$$\epsilon(\hat{\rho}(r)) = \exp\left(\frac{-k_o^2}{2}\Lambda\right), \tag{9.76a}$$

with

$$\Lambda = \int_0^X dr \int_0^X dr'\, B(|r - r'|, |y(r) - y(r')|, |z(r) - z(r')|). \tag{9.76b}$$

We now use Eq. (9.67), and Eq. (9.76b) becomes

$$\Lambda = \int_0^X dr \int_0^X dr'\, \zeta(|r - r'|)\, F(r, r'), \tag{9.77a}$$

where F is introduced to simplify the notation:

$$F(r, r') \equiv \tilde{B}(|y(r) - y(r')|, |z(r) - z(r')|). \tag{9.77b}$$

There is a standard technique for introducing sum and difference coordinates. Since the integrand is invariant under an interchange of r and r', we have

$$\Lambda = 2\int_0^X dr \int_0^r dr'\, \zeta(|r - r'|)\, F(r, r').$$

If we let $r' = r - s$,

$$\Lambda = 2\int_0^X dr \int_0^r ds\, \zeta(|s|)\, F(r, r - s)$$
$$= 2\int_0^X ds\, \zeta(|s|) \int_s^X dr\, F(r, r - s).$$

If we now let $r = t + s/2$, we have

$$\Lambda = 2\int_0^\infty ds\, \zeta(|s|) \int_{s/2}^{X - s/2} dt\, F\left(t + \frac{s}{2}, t - \frac{s}{2}\right). \tag{9.78}$$

We have replaced the upper limit on the integration over s by ∞, since we always assume $X \gg L_x$, and $\zeta(|s|) \simeq 0$ for $s \geq L_x$. Assuming the Markov approximation, we have

$$\Lambda = 2\int_0^\infty ds\, \delta(s) \int_0^X dt\, F(t, t).$$

But $F(t, t) = \tilde{B}(0, 0) = L_x\, \sigma_U^2$ and $\int_0^\infty ds\, \delta(s) = \frac{1}{2}$, so

$$\Lambda = X L_x \sigma_U^2. \tag{9.79}$$

We now relax the δ-function assumption and assume $\zeta(|s|)$ is only sharply peaked about $s = 0$. We then can examine the integral over t in Eq. (9.78) in a power series about $s = 0$. We shall keep terms through s^2. We first observe that

$$F\left(t + \frac{s}{2}, t - \frac{s}{2}\right) = \tilde{B}\left(\left|s \frac{dy}{dt} + O(s^3)\right|, \left|s \frac{dz}{dt} + O(s^3)\right|\right)$$
$$= \tilde{B}(0, 0)\left[1 - \frac{1}{2}s^2\left(\frac{1}{Ly}\frac{dy}{dt}\right)^2 - \frac{1}{2}s^2\left(\frac{1}{Lz}\frac{dz}{dt}\right)^2\right] + O(s^4). \tag{9.80}$$

Hence

$$\int_{s/2}^{X - s/2} dt\, F\left(t + \frac{s}{2}, t - \frac{s}{2}\right) = X L_x \sigma_U^2\left(1 - \frac{s}{X} - \frac{1}{2}s^2\gamma\right) + O(s^3), \tag{9.81}$$

where γ is the functional

$$\gamma = \frac{1}{L_y^2}\frac{1}{X}\int_0^X dr\left(\frac{dy}{dr}\right)^2 + \frac{1}{L_z^2}\frac{1}{X}\int_0^X dr\left(\frac{dz}{dr}\right)^2. \tag{9.82}$$

Substituting Eq. (9.81) into the expression for Λ gives

$$\Lambda = 2XL_x\, \sigma_U^2\left[\int_0^\infty ds\, \zeta(|s|)\left(1 - \frac{s}{X}\right) - \frac{1}{2}\gamma\int_0^\infty ds\, \zeta(|s|)\, s^2\right]. \tag{9.83}$$

The first integral is 1/2 since the s/R term may be dropped for $X \gg L_x$. The second integral is $L_x^2/2$ according to Eq. (9.70). Hence

$$\Lambda = XL_x \sigma_U^2 \left(1 - \frac{1}{2} L_x^2 \gamma\right). \tag{9.84}$$

This expression gives the first-order correction to the Markov approximation. By comparing Eqs. (9.79) and (9.84), we see we already have an interesting result. The Markov approximation is valid provided $L_x^2 \gamma \ll 1$ that is, provided

$$\frac{1}{X} \int_0^X dr \left(\frac{dy}{dr}\right)^2 \ll \left(\frac{L_y}{L_x}\right)^2 \tag{9.85a}$$

and

$$\frac{1}{X} \int_0^X dr \left(\frac{dz}{dr}\right)^2 \ll \left(\frac{L_z}{L_x}\right)^2. \tag{9.85b}$$

The Markov approximation implies the slopes of the Feynman paths, averaged over the range of propagation in an rms sense, must be small [5]. In the ocean where $L_x \simeq L_y \simeq 100\, L_z$, Eq. (9.85b) is far more stringent than Eq. (9.85a). Returning to Eqs. (9.76), we have

$$\epsilon(\hat{\rho}(r)) = \exp\left[\frac{-XL_x k_o^2 \sigma_U^2}{2}\left(1 - \frac{1}{2} L_x^2 \gamma\right)\right]$$

Substituting this expression into the path integral gives

$$\begin{aligned} \langle \Psi \rangle = \exp\left(\frac{-XL_x k_o^2 \sigma_U^2}{2}\right) \int D[\hat{\rho}(r)] \exp\Bigg[\frac{ik_o}{2}(1 - i\alpha_y) \int_0^X dt \left(\frac{dy}{dt}\right)^2 \\ + \frac{ik_o}{2}(1 - i\alpha_z) \int_0^X dt \left(\frac{dz}{dt}\right)^2\Bigg] \exp\left[-ik_o \int_0^X dr\, U_d(r, \hat{\rho}(r))\right], \end{aligned} \tag{9.86}$$

where

$$\alpha_y = \frac{k_o}{2} \frac{L_x^3}{L_y^2} \sigma_U^2 \tag{9.87a}$$

and

$$\alpha_z = \frac{k_o}{2} \frac{L_x^3}{L_z^2} \sigma_U^2. \tag{9.87b}$$

Equation (9.86) cannot be approximated without making assumptions about the deterministic sound speed. The fact that the Markov approximation is intimately related to the properties of the deterministic sound speed follows from Eqs. (9.85). The most important Feynman paths will be the deterministic paths calculated using ray acoustics. Therefore the magnitudes of the left-hand sides of Eqs. (9.85) are measures of the refractive characteristics of the deterministic sound speed. Rather than analyzing Eq. (9.86), which, in fact, is not well defined, we shall adopt a rather naive approach. Since $\alpha_x = \alpha_y = 0$ in the Markov approximation, we shall take as our validity conditions $\alpha_y \ll 1$ and $\alpha_z \ll 1$, that is

$$k_o L_x \sigma_U^2 \ll \min\left\{\left(\frac{L_y}{L_x}\right)^2, \left(\frac{L_z}{L_x}\right)^2\right\}. \tag{9.88}$$

For $L_x = L_y = L_z = L$ this reduces to the standard result

$$k_o L \sigma_U^2 \ll 1. \tag{9.89}$$

If we assume $U_d = 0$ and the Markov approximation is valid, Eq. (9.86) becomes

$$< \Psi > = \frac{k_o}{2\pi i X} \exp\left[\frac{i k_o X}{2}\left(\frac{\hat{\rho} - \hat{\rho}_s}{X}\right)^2\right] \exp\left(\frac{-X L_x k_o^2 \sigma_U^2}{2}\right). \tag{9.90}$$

In this case Eq. (9.53) gives

$$< p(\mathbf{x}) > = \frac{1}{4\pi X} \exp\left\{ i k_o \left[X + \frac{1}{2} \frac{(\hat{\rho} - \hat{\rho}_s)^2}{X} \right] \right\} \exp(-X/L_D), \tag{9.91}$$

where L_D is the *damping distance*:

$$\frac{1}{L_D} = \frac{L_x k_o^2 \sigma_U^2}{2}. \tag{9.92}$$

From Eq. (9.88) we see the Markov approximation requires that L_D be large in comparison to the acoustic wavelength.

Having discussed the Markov approximation, we are now in a position to discuss the paper by Klyatskin and Tatarskii. The following approach is adopted in Ref. 48: First, the mean index of refraction is taken to be a constant. This corresponds to setting $U_d(\mathbf{x})$ equal to zero. Second, a point source is not considered. Instead, it is assumed that a field with some complex amplitude distribution $u_o(\hat{\rho})$ is incident on the $x = 0$ plane and that the random perturbations in the sound speed exist only in the half-space $x > 0$. Third, the path-integral representation given by Eq. (8.20) is used. Fourth, the fluctuations in the index of refraction are homogeneous and isotropic. Fifth, although it is not necessary to introduce a correlation length, validity conditions are recorded in terms of such a quantity. Sixth, the Markov approximation is assumed. Seventh, the average field and the mutual coherence function are considered. Only the results of the calculation of the mutual coherence function are recorded.

In our analysis we shall assume: the mean value U_d is zero (the general case will be considered in the following subsections), the radiation is due to a point source, the model outlined above describes the fluctuations, and the Markov approximation is valid. Moreover, we will calculate only the average field, and we will use the path-integral representation we listed as Eqs. (9.10) through (9.15).

Returning to Eqs. (9.10) and (9.11), we have

$$< p(\mathbf{x}) > = \frac{i}{2 k_o} \int_0^\infty d\tau \exp\left(\frac{i k_o \tau}{2}\right) < \Phi(\tau, \mathbf{x} | 0, \mathbf{x}_s) >. \tag{9.93}$$

Just as before, we redefine the paths:

$$x(\sigma) = x_s + \frac{\sigma}{\tau}(x - x_s) + x'(\sigma),$$
$$y(\sigma) = y'(\sigma),$$

and

$$z(\sigma) = z'(\sigma). \tag{9.94}$$

Then

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \exp\left[\frac{ik_o}{2\tau} X^2\right] \int D[\mathbf{x}'(\sigma)] \exp\left\{ik_o \int_0^\tau d\sigma \left[\frac{1}{2}\left(\frac{d\mathbf{x}'}{d\sigma}\right)^2 - U\left(x_s + \frac{\sigma}{\tau} X + x'(\sigma), \hat{\rho}(\sigma)\right)\right]\right\}, \tag{9.95}$$

where again $X = x - x_s$. Taking the average yields

$$< \Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) > = \exp\left[\frac{ik_o}{2\tau} X^2\right] \int D[\mathbf{x}(\sigma)] \times \exp\left[\frac{ik_o}{2} \int_0^\tau d\sigma \left(\frac{d\mathbf{x}}{d\sigma}\right)^2 - \frac{1}{2} k_o^2 \Lambda\right], \tag{9.96}$$

with

$$\Lambda = \int_0^\tau d\sigma \int_0^\tau d\sigma' \zeta\left\{\left|\frac{\sigma}{\tau} X - \frac{\sigma'}{\tau} X + x(\sigma) - x(\sigma')\right|\right\} F(\sigma, \sigma'),$$

in which

$$F(\sigma, \sigma') = \tilde{B}(|y(\sigma) - y(\sigma')|, |z(\sigma) - z(\sigma')|). \tag{9.97}$$

The prime on the path has been dropped in writing these expressions. We need only remember that now we have $x(0) = x(\tau) = 0$. Following the steps which led to Eq. (9.78), we get

$$\Lambda = 2 \int_O^\tau ds \int_{s/2}^{\tau - s/2} dt\, \zeta\left(\left|\frac{X}{\tau} s + x\left(t + \frac{s}{2}\right) - x\left(t - \frac{s}{2}\right)\right|\right) F\left(t + \frac{s}{2}, t - \frac{s}{2}\right). \tag{9.98}$$

If we were to apply straight-line geometric optics to the x coordinate, $x(t + s/2) - x(t - s/2)$ would be dropped. Klyatskin and Tatarskii go one step further and calculate Λ to first order in this difference:

$$\zeta\left[\left|\frac{X}{\tau} s + x\left(t + \frac{s}{2}\right) - x\left(t - \frac{s}{2}\right)\right|\right] = \zeta\left(\left|\frac{Xs}{\tau}\right|\right) + \left[x\left(t + \frac{s}{2}\right) - x\left(t - \frac{s}{2}\right)\right] \zeta'\left(\left|\frac{Xs}{\tau}\right|\right) + \ldots \tag{9.99}$$

With the Markov approximation, Eq. (9.71), this last equation reads

$$\zeta = \frac{\tau}{X} \delta(s) + \frac{\tau}{X} \delta'(s) \left[x\left(t + \frac{s}{2}\right) - x\left(t - \frac{s}{2}\right)\right] + \ldots \tag{9.100}$$

Therefore

$$\Lambda = \Lambda_1 + \Lambda_2 + \ldots, \tag{9.101a}$$

where

$$\Lambda_1 = \frac{2\tau}{X} \int_0^{\tau} ds\, \delta(s) \int_{s/2}^{\tau - s/2} dt\, F\left(t + \frac{s}{2}, t - \frac{s}{2}\right) \tag{9.101b}$$

and

$$\Lambda_2 = \frac{2\tau}{X} \int_0^{\tau} ds\, \delta'(s) \int_{s/2}^{\tau - s/2} dt \left[x\left(t + \frac{s}{2}\right) - x\left(t - \frac{s}{2}\right)\right] F\left(t + \frac{s}{2}, t - \frac{s}{2}\right). \tag{9.101c}$$

The function Λ_1 represents the result of applying straight-line geometric optics to the x coordinate, and Λ_2 is the first-order correction to this approximation. It is not difficult to show

$$\Lambda_1 = \frac{\tau^2}{X} L_x \sigma_U^2 \tag{9.102a}$$

and

$$\Lambda_2 = \frac{\tau}{X} L_x \sigma_U^2 \int_0^{\tau} ds\, s\, \delta'(s) \int_0^{\tau} dt \left(\frac{dx}{dt}\right) = 0. \tag{9.102b}$$

The right-hand of this last equality follows because $x(0) = x(\tau) = 0$. Therefore *the first-order correction to straight-line geometric optics vanishes if the Markov approximation is valid.*

We now drop higher order corrections to straight-line geometric optics, indicated by the dots in Eq. (9.101a), and substitute Eqs. (9.102a) and (9.102b) back into Eq. (9.96):

$$<\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s)> = \exp\left[\frac{ik_o}{2\tau} X^2 - \frac{k_o^2 \tau^2 L_x \sigma_U^2}{2X}\right] \int D[\mathbf{x}(\sigma)] \exp\left[\frac{ik_o}{2} \int_0^{\tau} d\sigma \left(\frac{d\mathbf{x}}{d\sigma}\right)^2\right]$$

or

$$<\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s)> = \left(\frac{k_o}{2\pi i \tau}\right)^{3/2} \exp\left[\frac{ik_o}{2\tau} |\mathbf{x} - \mathbf{x}_s|^2 - \frac{\tau^2}{XL_D}\right]. \tag{9.103}$$

In writing this equation, we have used

$$|\mathbf{x} - \mathbf{x}_s|^2 = X^2 + (\hat{\rho} - \hat{\rho}_s)^2 \tag{9.104}$$

and Eq. (9.92), the definition for L_D. Equation (9.93) becomes

$$<p(\mathbf{x})> = \frac{1}{4\pi} \left(\frac{k_o}{2\pi i}\right)^{1/2} \int_0^{\infty} \frac{d\tau}{\tau^{3/2}} \exp\left[\frac{ik_o}{2}\left(\tau + \frac{|\mathbf{x} - \mathbf{x}_s|^2}{\tau}\right) - \frac{\tau^2}{XL_D}\right]. \tag{9.105}$$

An integral of the form

$$I = \frac{1}{4\pi} \left(\frac{k_o}{2\pi i}\right)^{1/2} \int_0^{\infty} \frac{d\tau}{\tau^{3/2}} \exp\left[\frac{ik_o}{2}\left(\tau + \frac{A^2}{\tau}\right) + f(\tau)\right] \tag{9.106}$$

may be written as

$$I = \frac{1}{4\pi A} \int_{-\infty}^{\infty} d\sigma \exp f(\sigma) \int_{-\infty}^{\infty} \frac{dq}{(2\pi)} \exp\left(-iq\sigma + iAk_o \sqrt{1 + 2q/k_o}\right). \tag{9.107}$$

By using the expansion

$$\sqrt{1 + 2q/k_o} = 1 + q/k_o - \frac{1}{2} q^2/k_o^2 + \ldots \tag{9.108}$$

we have

$$I \simeq \frac{\exp(iAk_o)}{4\pi A}\left[\frac{k_o}{2\pi iA}\right]^{1/2}\int_{-\infty}^{\infty} d\sigma \exp\left[f(\sigma) + \frac{ik_o(\sigma - A)^2}{2A}\right]. \tag{9.109}$$

With the aid of this last expression, Eq. (9.105) becomes

$$\langle p(\mathbf{x})\rangle = \frac{\exp ik_o|\mathbf{x} - \mathbf{x}_s|}{4\pi|\mathbf{x} - \mathbf{x}_s|}\left(1 + \frac{2i|\mathbf{x} - \mathbf{x}_s|}{k_oXL_D}\right)^{-1/2}$$
$$\times \exp\left[-\frac{(\mathbf{x} - \mathbf{x}_s)^2}{XL_D}\left(1 + \frac{2i|\mathbf{x} - \mathbf{x}_s|}{k_oXL_D}\right)^{-1}\right]. \tag{9.110}$$

This expression is to be compared with $\langle p(\mathbf{x})\rangle$ calculated assuming the parabolic approximation, Eq. (9.91):

$$\langle p(\mathbf{x})\rangle = \frac{\exp[ik_oX + ik_o(\hat{\rho} - \hat{\rho}_s)^2/2X]}{4\pi X}\exp(-X/L_D). \tag{9.111}$$

The two expressions agree if

$$\left(\frac{\hat{\rho} - \hat{\rho}_s}{X}\right)^2 \ll 1, \tag{9.112a}$$

$$\frac{k_o X}{8}\left(\frac{\hat{\rho} - \hat{\rho}_s}{X}\right)^4 \ll 1, \tag{9.112b}$$

$$k_o L_D \gg 1, \tag{9.113a}$$

and

$$k_o L_D^2 \gg X. \tag{9.113b}$$

Equations (9.112) are simple kinematic restrictions, and Eq. (9.113a) is the condition we previously found for the validity of the Markov approximation.

In summary, *the condition for the validity of the Markov approximation is the same as one of the conditions for the validity of the parabolic approximation.* Moreover, with reference to Eqs. (9.85), *the parabolic approximation implies the slopes of the Feynman paths must on the average be small.*

9.4 Relaxing the Straight-Line Geometric-Optics Approximation: A Modified Parabolic Equation

Having discussed the calculation by Klyatskin and Tatarskii, we now consider corrections to the parabolic approximation in the general case in which the sound speed is an arbitrary function of $\mathbf{x}$. In this subsection the straight-line geometric-optics approximation will be relaxed, and in the next subsection we will discuss corrections to the stationary-phase approximation. Rather than decomposing U into deterministic and random components and considering the average value of the field, we will use the unaveraged expressions developed in subsections 9.1 and 9.2. Moreover we consider only the two-dimensional parabolic equation.

Horizontal straight-line geometric optics amounted to assuming that in Eq. (9.20)

$$U\left[s_\perp + \frac{\sigma R}{\tau} e_\perp + x_\perp'(\sigma), z'(\sigma)\right] \simeq U\left[s_\perp + \frac{\sigma R}{\tau} e_\perp, z'(\sigma)\right]. \tag{9.114}$$

This approximation may be relaxed in several ways. We could take

$$U\left[s_\perp + \frac{\sigma R}{\tau} e_\perp + x_\perp'(\sigma), z'(\sigma)\right] \simeq U\left[s_\perp + \frac{\sigma R}{\tau} e_\perp, z'(\sigma)\right] + x_\perp'(\sigma)\cdot\nabla_\perp U\left[s_\perp + \frac{\sigma R}{\tau} e_\perp, z'(\sigma)\right] \tag{9.115}$$

and evaluate the integral over $x_\perp'(\sigma)$, using an equation similar to Eq. (6.71). We would end up with a two-dimensional parabolic equation with U replaced by an expression involving integrals over the path parameter r of U and U^2. Here we would like to discuss a different, more general approach. Rather than making a horizontal straight-line geometric-optics approximation, we will make a *horizontal Rytov approximation.*

Equation (9.11) may be written in the form

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \int D[z(\sigma)] \exp\left[\frac{ik_o}{2}\int_0^\tau d\sigma \left(\frac{dz}{d\sigma}\right)^2\right]\Phi_\perp(\tau, x_\perp|0, s_\perp), \tag{9.116}$$

where

$$\Phi_\perp = \int D[x_\perp(\sigma)] \exp\left\{ik_o \int_0^\tau d\sigma \left[\frac{1}{2}\left(\frac{dx_\perp}{d\sigma}\right)^2 - U(x_\perp(\sigma), z(\sigma))\right]\right\}. \tag{9.117}$$

This integral is a functional of the path $z(\sigma)$. It can be evaluated using the Rytov approximation (subsection 6.4):

$$\Phi_\perp(\tau, x_\perp|0, s_\perp) = \Phi_{o\perp}(\tau, x_\perp|0, s_\perp) \exp\left[\frac{\text{Born term}}{\Phi_{o\perp}(\tau, x_\perp|0, s_\perp)}\right], \tag{9.118}$$

where

$$\text{Born term} = -ik_o \int_0^\tau d\sigma \int dx_\perp' \, \Phi_{o\perp}(\tau, x_\perp|\sigma, x_\perp)\, U(x_\perp' z(\sigma))\, \Phi_{o\perp}(\sigma, x_\perp'|0, s_\perp) \tag{9.119}$$

and

$$\Phi_{o\perp}(\tau, x_\perp|\tau', x_\perp') = \int D[x_\perp(\sigma)] \exp\left[\frac{ik_o}{2}\int_{\tau'}^\tau d\sigma\left(\frac{dx_\perp}{d\sigma}\right)^2\right] = \left(\frac{k_o}{2\pi i(\tau - \tau')}\right)\exp\left[\frac{i k_o(\tau - \tau')}{2}\left(\frac{x_\perp - x_\perp'}{\tau - \tau'}\right)^2\right]. \tag{9.120}$$

Substituting Eqs. (9.119) and (9.120) into Eq. (9.118) yields

$$\Phi_\perp(\tau, x_\perp|0, s_\perp) = \left(\frac{k_o}{2\pi i\tau}\right)\exp\left(\frac{ik_o R^2}{2\tau}\right)\exp\left\{-ik_o\int_0^\tau d\sigma\left[\frac{k_o \tau}{2\pi i\sigma(\tau - \sigma)}\right]\right\} \times \int dx_\perp' \, U(x_\perp', z(\sigma)) \exp\left\{\frac{i k_o \tau}{2\sigma(\tau - \sigma)}\left[s_\perp - x_\perp' + \frac{\sigma}{\tau}(x_\perp - s_\perp)\right]^2\right\}. \tag{9.121}$$

With Eq. (9.121), Eq. (9.116) becomes

$$\Phi(\tau, \mathbf{x}|0, \mathbf{x}_s) = \left(\frac{k_o}{2\pi i\tau}\right) \exp\left(\frac{ik_o R^2}{2\tau}\right) \tilde{\Gamma}(\tau, z|0, z_s), \tag{9.122}$$

where

$$\tilde{\Gamma}(\tau, z|0, z_s) = \int D[z(\sigma)] \exp\left\{i\, k_o \int_0^\tau d\sigma \left[\frac{1}{2}\left(\frac{dz}{d\sigma}\right)^2 - U_{eff}(\sigma, \tau, z(\sigma))\right]\right\}, \tag{9.123}$$

with

$$U_{eff}(\sigma, \tau, z(\sigma)) = \frac{k_o \tau}{2\pi i\sigma(\tau - \sigma)} \int dx_\perp'\, U(x_\perp', z(\sigma))$$

$$\exp\left[\frac{i\, k_o\, \tau}{2\, \sigma(\tau - \sigma)}\left(s_\perp - x_\perp' + \frac{\sigma}{\tau} R\, e_\perp\right)^2\right]. \tag{9.124}$$

These equations are to be compared with Eqs. (9.21) and (9.22).

Since the action in Eq. (9.123) is not additive, we generalize (imbed) just as before by introducing

$$\tilde{\Gamma}_\tau'(t, z|0, z_s) = \int D[z(\sigma)] \exp\left\{ik_o \int_0^t d\sigma \left[\frac{1}{2}\left(\frac{dz}{d\sigma}\right)^2 - U_{eff}(\sigma, \tau, z(\sigma))\right]\right\}. \tag{9.125}$$

We have

$$\tilde{\Gamma}_\tau'(\tau, z|0, z_s) = \tilde{\Gamma}(\tau, z|0, z_s) \tag{9.126a}$$

and

$$-2\, ik_o\, \partial_t\, \tilde{\Gamma}_\tau'(t, z|0, z_s) = [\partial_z^2 - 2\, k_o^2\, U_{eff}(t, \tau, z)]\, \tilde{\Gamma}_\tau'(t, z|0, z_s), \tag{9.126b}$$

where

$$\tilde{\Gamma}_\tau'(0, z|0, z_s) = \delta(z - z_s). \tag{9.126c}$$

In analogy with Eqs. (9.35) we define

$$\tilde{\Psi}_\tau(r, z|0, z_s) = \tilde{\Gamma}_\tau'\left(\frac{\tau r}{R}, z|0, z_s\right) \tag{9.127}$$

and obtain the system of equations

$$p(\mathbf{x}) = \frac{1}{4\pi} \int_0^\infty \frac{d\tau}{\tau} \exp\left[\frac{i\, k_o}{2}\left(\tau + \frac{R^2}{\tau}\right)\right] \tilde{\Psi}_\tau(R, z|0, z_s), \tag{9.128a}$$

where

$$\frac{-2\, i\, k_o}{(\tau/R)}\, \partial_r\, \tilde{\Psi}_\tau(r, z|0, z_s) = [\partial_z^2 - 2k_o^2\, U_{eff}^\tau(r, z)]\, \tilde{\Psi}_\tau(r, z|0, z_s), \tag{9.128b}$$

in which

$$\tilde{\Psi}_\tau (0, z|0, z_s) = \delta (z - z_s), \tag{9.128c}$$

and

$$U^\tau_{eff}(r,z) = \frac{k_o R (R/\tau)}{2 \pi ir(R - r)} \int dx_\perp' \, U(x_\perp', z) \exp\left[\frac{i k_o R (R/\tau)}{2r(R - r)} (s_\perp - x_\perp' + r e_\perp)^2\right]. \tag{9.128d}$$

If the integral over τ in Eq. (9.128a) is evaluated by the method of stationary phase, we have

$$p(\mathbf{x}) = \frac{1}{4 \pi} \left(\frac{2 \pi i}{k_o R}\right)^{1/2} e^{ik_o R} \, \tilde{\Psi} (R, z|0, z_s), \tag{9.129a}$$

with

$$-2i \, k_o \, \partial_r \tilde{\Psi} (r, z|0, z_s) = \{\partial_z^2 + k_o^2 [n^2_{eff}(r, z) - 1]\} \tilde{\Psi} (r, z|0, z_s), \tag{9.129b}$$

$$\tilde{\Psi} (0, z|0, z_s) = \delta (z - z_s), \tag{9.129c}$$

and

$$n^2_{eff}(r, z) = \frac{k_o R}{2 \pi ir(R - r)} \int d x_\perp' \, n^2(x_\perp', z) \exp\left[\frac{i k_o R}{2r(R - r)} (s_\perp - x_\perp' + r e_\perp)^2\right]. \tag{9.129d}$$

This system of equations was first derived in Ref. 84 using different techniques. Equation (9.129b) is identical in form to the usual two-dimensional parabolic equation with the exception that the index of refraction is replaced by an effective index of refraction obtained by integrating over the horizontal coordinates. The dominant contributions to this two-dimensional integral come from a cigar-shaped region about the geometric-optics path. The extent of this region is determined by the horizontal gradients of $n^2(\mathbf{x})$ and typically is of the order of 10 kms. This impoved equation is not particularly useful when modeling the effects of sound-speed fluctuations which are horizontally homogeneous and isotropic. It does offer an advantage, however, if one is interested in modeling the effects of large-scale anomalies such as eddies or fronts, because it allows one to incorporate into a two-dimensional parabolic equation three-dimensional variations in the sound speed.

9.5 Corrections to the Stationary-Phase Approximation

In calculating corrections to the stationary-phase approximation, we shall assume that the horizontal straight-line geometric optics approximation is valid. The extension to the modified parabolic equation developed in the preceeding subsection is straight forward. We consider therefore Eqs. (9.36).

Since the stationary-phase approximation amounts to the assumption that Ψ_τ is slowly varying about the stationary-phase point $\tau = R$, corrections may be obtained by expanding Ψ_τ about $\tau = R$:

$$\Psi_\tau (R, z|0, z_s) = \sum_{l=0}^{\infty} \frac{1}{l!} (\tau - R)^l \, \Psi^{(l)} (R, z|0, z_s), \tag{9.130a}$$

where

$$\Psi^{(l)}(R, z|0, z_s) = \partial_\tau^l \Psi_\tau(R, z|0, z_s)|_{\tau=R}. \tag{9.130b}$$

Substituting Eq. (9.130a) into Eq. (9.36a) gives

$$p(\mathbf{x}) = \frac{1}{4\pi}\left(\frac{2\pi i}{k_o R}\right)^{1/2} e^{ik_o R} \sum_{l=0}^{\infty} \frac{R^l}{l!} I_l(k_o R)\,\Psi^{(l)}(R, z|0, z_s), \tag{9.131}$$

with

$$I_l(z) = \left(\frac{z}{2\pi i}\right)^{1/2} e^{-iz} \int_0^\infty \frac{dt}{t}(t-1)^l \exp\left[\frac{iz}{2}(t + t^{-1})\right]. \tag{9.132}$$

The calculation of corrections to the stationary phase approximation reduces to the calculation of the integrals I_l and the derivatives $\Psi^{(l)}$.

We first indicate how I_l may be calculated for any l. By the change of variable $t \to t^{-1}$ it is possible to write

$$I_l(z) = \frac{N}{2}\int_0^\infty \frac{dt}{t} \exp\left[\frac{iz}{2}(t + t^{-1})\right]\left[(t-1)^l + \left(\frac{1}{t} - 1\right)^l\right], \tag{9.133a}$$

where N is the coefficient in Eq. (9.132):

$$N = \left(\frac{z}{2\pi i}\right)^{1/2} e^{-iz}. \tag{9.133b}$$

For any l the expression in the second pair of brackets in Eq. (9.133) may be written as an lth order polynominal in $t + (1/t)$:

$$\left[(t-1)^l + \left(\frac{1}{t} - 1\right)^l\right] = \sum_{m=0}^{l} A_m^l \left(t + \frac{1}{t}\right)^m.$$

With this expression Eq. (9.133) becomes

$$I_l(z) = \frac{N}{2}\sum_{m=0}^{l}\left(\frac{2}{i}\right)^m A_m^l\, \partial_z^m \int_0^\infty \frac{dt}{t}\exp\left[\frac{iz}{2}(t + t^{-1})\right]. \tag{9.134}$$

Equation (A12) gives

$$I_l(z) = \frac{i\pi}{2} N \sum_{m=0}^{l}\left(\frac{2}{i}\right)^m A_m^l \frac{d^m}{dz^m} H_o^{(1)}(z). \tag{9.135}$$

Since the Hankel function obeys Bessel's equation

$$\left(\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + 1\right) H_o^{(1)}(z) = 0,$$

it is always possible to write

$$\begin{aligned}\frac{d^m}{dz^m} H_o^{(1)}(z) &= b_m H_o^{(1)}(z) - c_m \frac{d}{dz} H_o^{(1)}(z),\\ &= b_m H_o^{(1)}(z) + c_m H_1^{(1)}(z),\end{aligned}$$

thus expressing all the higher order derivatives in terms of the first and second derivatives. The expression for I_l now becomes

$$I_l(z) = \frac{i\pi}{2} N \sum_{m=0}^{l} \left(\frac{2}{i}\right)^m A_m^l \left[b_m H_o^{(1)}(z) + c_m H_1^{(1)}(z)\right]. \quad (9.136)$$

This expression gives the exact value of the integral. The determination of the coefficients a_m^l, b_m, and c_m is straightforward for any l and m (but tedius for large l or m). As an illustration we have, for $l \leqslant 2$,

$$A_0^0 = 2 \qquad A_0^1 = -2 \qquad A_0^2 = 0$$
$$A_1^1 = 1 \qquad A_1^2 = -2$$
$$A_2^2 = 1$$

and

$$b_0 = 1 \qquad b_1 = 0 \qquad b_2 = -1$$
$$c_0 = 0 \qquad c_1 = -1 \qquad c_2 = 1/z,$$

giving

$$I_0(z) = i\pi N H_o^{(1)}(z), \quad (9.137a)$$

$$I_1(z) = -i\pi N\left[1 + i\frac{d}{dz}\right] H_o^{(1)}(z), \quad (9.137b)$$

and

$$I_2(z) = 2i\pi N\left[1 + \left(i + \frac{1}{z}\right)\frac{d}{dz}\right] H_o^{(1)}(z). \quad (9.137c)$$

Since $z = k_o R$ is always large, it is usually sufficient to approximate the Hankel function by the first term of its asymptotic series:

$$i\pi H_o^{(1)}(z) \simeq N^{-1}. \quad (9.138)$$

Substituting Eq. (9.138) into Eqs. (9.137) and carrying out the indicated differentiations gives

$$I_0(z) = 1, \quad (9.139a)$$

$$I_1(z) = -\frac{1}{2iz}, \quad (9.139b)$$

and

$$I_2(z) = -\frac{1}{iz} + \frac{1}{(iz)^2}. \quad (9.139c)$$

Equations (9.130b), (9.131), (9.132), and (9.139) yield

$$p(\mathbf{x}) = \frac{1}{4\pi}\left(\frac{2\pi i}{k_o R}\right)^{1/2} e^{ik_o R}\left[\Psi(R, z|0, z_s) + \frac{iR}{2k_o}\,\partial_\tau^2 \Psi_\tau(R, z|0, z_s)|_{\tau=R} + \ldots\right]. \quad (9.140)$$

In writing this expression we have kept only the leading correction which increases with range.

Consider now the situation where the sound speed is a function of depth only. From Eq. (9.43) we obtain

$$\partial_\tau^2 \Psi_\tau (R, z|0, z_s)|_{\tau=R} = \partial_R^2 \Psi (R, z|0, z_s), \tag{9.141}$$

and Eq. (9.140) becomes

$$p(\mathbf{x}) = \frac{1}{4\pi} \left(\frac{2\pi i}{k_o R} \right)^{1/2} e^{ik_o R} \Psi (R, z|0, z_s) \; (1 + \Delta), \tag{9.142}$$

where Δ is an estimate of the error associated with the stationary-phase approximation:

$$\Delta = \frac{iR}{2k_o} \left[\frac{\partial_R^2 \Psi (R, z|0, z_s)}{\Psi (R, z|0, z_s)} \right]. \tag{9.143}$$

This is as far as one can go without specifying the sound speed and using a model solution to calculate Δ via

$$\Psi (R, z|0, z_s) = \sum_n Z_n (z) \; Z_n (z_s) \exp \left[\frac{iR}{2k_o} (k_n^2 - k_o^2) \right]. \tag{9.144}$$

(This equation is to be compared with Eq. (9.42).) For a range-independent sound speed the straight-line geometric-optics approximation is exact. Therefore the stationary-phase approximation is equivalent to the parabolic approximation, and the question of the validity of the parabolic approximation reduces to the question of the size of Δ. In this case one does not need to numerically solve the parabolic equation in order to discuss the error. All that is required is values for the normal-mode eigenfunctions and eigenvalues.

If the sound speed possess a range dependence, that is, if it depends on the horizontal coordinates $x_\perp = (x, y)$, then Eq. (9.141) is no longer valid. The breakdown of Eq. (9.141) was suggested when we wrote Eq. (9.46) and may be inferred from Eq. (9.31). We could calculate the correction to the stationary-phase approximation directly from Eq. (9.140) by numerically solving for $\Psi_\tau(R, z|0, z_s)$ for a range of values of τ about $\tau = R$ and then numerically computing the second derivative. For a typical range-dependent sound speed, however, the most important correction to the stationary-phase approximation (not the parabolic approximation) is again given by Eqs. (9.142) and (9.143) [85]. It would be desirable however to have a criterion which determines when Eqs. (9.142) and (9.143) are likely to be modified as the result of the range dependence of the sound speed.

It is not difficult to show the solution to Eq. (9.36b) satisfies the integral equation

$$\begin{aligned} \Psi_\tau (r, z|0, z_s) = \Psi_\tau^{(0)} (r, z|0, z_s) - \frac{ik_o\tau}{R} \int_0^r dr_1 \int dz_1 \\ \times \Psi_\tau^{(0)} (r, z|r_1, z_1) \; U(r_1, z_1) \; \Psi_\tau (r_1, z_1|0, z_s). \end{aligned} \tag{9.145}$$

Here $\Psi_\tau^{(0)}$ satisfies Eq. (9.36b) with $U = -\frac{1}{2}(n^2 - 1)$ set equal to zero. Since

$$\Psi_{tR}^{(0)} (r, z|r_1, z_1) = \Psi_R^{(0)} (tr, z|tr_1, z_1), \tag{9.146}$$

it follows that

$$\begin{aligned} \Psi_{tR} (r, z|0, z_s) = \Psi_R^{(0)} (tr, z|0, z_s) - ik_o \int_0^{tr} dr_1 \int dz_1 \\ \times \Psi_R^{(0)} (tr, z|r_1, z_1) \; U\left(\frac{r_1}{t}, z_1 \right) \Psi_{tR} (r_1, z_1|0, z_s). \end{aligned} \tag{9.147}$$

Therefore

$$\Psi_{tR}(R, z|0, z_s) = \Omega^t(tR, z|0, z_s), \tag{9.148}$$

where Ω is defined by the equations

$$\left[2ik_o\partial_r + \partial_z^2 - 2k_o U\left(\frac{r}{t}, z\right)\right] \Omega^t(r, z|0, z_s) = 0 \tag{9.149}$$

and

$$\Omega^t(0, z|0, z_s) = \delta(z - z_s). \tag{9.150}$$

The function Ω depends on t in two distinct ways. First, t scales the range. This dependence persists even when the sound speed is independent of the horizontal coordinates. Second, Ω has a t dependence which results solely from the range dependence of the sound speed.

In analogy with Eq. (9.131) we have

$$p(\mathbf{x}) = \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{R^l}{l!} \int_0^{\infty} \frac{dt}{t} (t-1)^l \exp\left[\frac{ik_oR}{2}\left(t + \frac{1}{t}\right)\right] \partial_R^l \,\Omega^t(R, z|0, z_s). \tag{9.151}$$

If we take

$$\Omega^t(R, z|0, z_s) \simeq \Omega^{t=1}(R, z|0, z_s) = \Psi(R, z|0, z_s), \tag{9.152}$$

Eq. (9.151) reduces to Eq. (9.140) (for k_oR large). In developing a criterion which will determine when Eqs. (9.142) and (9.143) are valid, it is sufficient to consider only the $l = 0$ term in Eq. (9.151):

$$\frac{1}{4\pi} \int_0^{\infty} \frac{dt}{t} \exp\left[\frac{ik_oR}{2}\left(t + \frac{1}{t}\right)\right] \Omega^t(R, z|0, z_s). \tag{9.153}$$

(If the lowest order term is in error, there is little hope for the higher order terms.) With the stationary-phase approximation, Eq. (9.153) reduces to Eq. (9.47) because of Eq. (9.152).

We now write a path integral for Ω:

$$\Omega^t(R, z|0, z_s) = \int D[z(r)] \exp\left\{ik_o \int_0^R dr\left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 - U\left(\frac{r}{t}, z(r)\right)\right]\right\}. \tag{9.154}$$

The t dependence of Ω occurs only in the argument of U. Therefore the term labeled Eq. (9.153) may be rewritten in the form

$$\int D[z(r)] \exp\left[\frac{ik_o}{2} \int_0^R dr\left(\frac{dz}{dr}\right)^2\right] J[z(r)], \tag{9.155}$$

where

$$J[z(r)] = \frac{1}{4\pi} \int_0^{\infty} \frac{dt}{t} \exp\left[\frac{ik_oR}{2}\left(t + \frac{1}{t}\right) - ik_o \int_0^R dr\, U\left(\frac{r}{t}, z(r)\right)\right]. \tag{9.156}$$

By carrying out an analysis similar to that which gave Eq. (9.110), we find

$$J[z(r)] \simeq \frac{1}{4\pi}\left(\frac{2\pi i}{k_o R}\right)^{1/2} \exp\left[ik_o R - ik_o \int_0^R dr\, U(r, z(r))\right] \exp\left\{\frac{-i}{2k_o R}(F[z(r)])^2\right\}, \tag{9.157}$$

where

$$F[z(r)] = k_o \int_0^R dr\, rU'(r, z(r)), \tag{9.158a}$$

with

$$U'(r, z) \equiv \partial_r\, U(r, z). \tag{9.158b}$$

Substituting Eq. (9.157) into Eq. (9.155) yields

$$\frac{1}{4\pi}\left(\frac{2\pi i}{k_o R}\right)^{1/2} e^{ik_o R} \int D[z(r)] \exp\left\{ik_o \int_0^R dr\left[\frac{1}{2}\left(\frac{dz}{dr}\right)^2 - U(r, z(r))\right]\right\} \exp\left(\frac{-iF^2}{2k_o R}\right). \tag{9.159}$$

Equation (9.159) agrees with Eq. (9.47) provided $F^2/2k_o R$ is small, say a quarter of a cycle:

$$\frac{F^2}{2k_o R} < \frac{\pi}{2}$$

or

$$k_o\left|\int_0^R dr\, rU'(r, z(r))\right|^2 < \pi R. \tag{9.160}$$

(This expression is analogous to Eq. (9.113b).) To a good approximation Eq. (9.160) may be replaced by

$$k_o\left|\int_0^R dr\, rU'(r, z^*(r))\right|^2 < \pi R. \tag{9.161}$$

where $z^*(r)$ is any one of the ray paths defined by ray acoustics. Equation (9.161) gives an operational procedure for determining when the leading order correction to the stationary-phase approximation is not given simply by Eq. (9.143).

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APPENDIX A
THE FREE-SPACE GREEN'S FUNCTION IN n DIMENSIONS

Let $\bar{x} = (x_1, ..., x_n)$ be an n-dimensional coordinate vector, and let $\bar{\nabla}^2 = \sum_{i=1}^{n} \partial_{x_i}^2$ be the associated Laplacian. The n-dimensional, free-space Green's function $G_n(\bar{x}|\bar{x}')$ satisfies

$$(\bar{\nabla}^2 + k_o^2) G_n(\bar{x} - \bar{x}') = -\delta^{(n)}(\bar{x} - \bar{x}'), \tag{A1}$$

where $\delta^{(n)}(\bar{x})$ is the n-dimensional δ-function. In this appendix a one-parameter integral representation for G_n will be constructed.

Introducing the Fourier transform

$$G_n(\bar{x} - \bar{x}') = \int \frac{d^n\bar{k}}{(2\pi)^n} e^{i\bar{k}\cdot(\bar{x}-\bar{x}')} G(\bar{k}), \tag{A2}$$

we have

$$G(\bar{k}) = \frac{1}{\bar{k}^2 - k_o^2 - i\epsilon} \tag{A3}$$

for outgoing waves. We now use

$$\frac{1}{\bar{k}^2 - k_o^2 - i\epsilon} = i \int_0^\infty d\tau \, e^{-i\tau(\bar{k}^2 - k_o^2 - i\epsilon)}, \tag{A4}$$

which follows from the Fourier transform of the step function

$$\theta(\tau) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega\tau}}{\omega - i\epsilon}, \tag{A5}$$

where

$$\begin{aligned} \theta(\tau) &= 1, \quad \tau > 0, \\ &= 0, \quad \tau < 0. \end{aligned} \tag{A6}$$

Substituting Eqs. (A3) and (A4) into Eq. (A2) and interchanging the orders of integration gives

$$\begin{aligned} G_n(\bar{x} - \bar{x}') &= i \int_0^\infty d\tau \, e^{i\tau k_o^2} \int \frac{d^n\bar{k}}{(2\pi)^n} e^{i\bar{k}\cdot(\bar{x}-\bar{x}') - i\bar{k}^2\tau} \\ &= i \int_0^\infty d\tau \, e^{i\tau k_o^2} \prod_{j=1}^{n} \left[\int \frac{dk_j}{(2\pi)} e^{ik_j(x-x')_j - ik_j^2\tau} \right]. \end{aligned} \tag{A7}$$

The n-dimensional integral over $\bar{k}$ breaks up into n one-dimensional integrals, and each of these one-dimensional integrals can be readily evaluated (Eq. (4.21)) giving

$$G_n(\bar{x} - \bar{x}') = i \int_0^\infty d\tau \left(\frac{1}{4\pi i\tau} \right)^{n/2} e^{i\tau k_o^2 + iR_n^2/4\tau}, \tag{A8}$$

where R_n is the magnitude of $\bar{x} - \bar{x}'$:

$$R_n^2 = \sum_{j=1}^{n} (x_j - x_j')^2. \tag{A9}$$

With the change of variable

$$\tau \to \frac{R_n}{2k_o} \tau$$

we have

$$G_n(\bar{x} - \bar{x}') = \frac{1}{4\pi} \left[\frac{k_o}{2\pi i R_n}\right]^{(n/2)-1} \int_0^\infty \frac{d\tau}{\tau^{n/2}} \exp\left[\frac{ik_o R_n}{2}\left(\tau + \frac{1}{\tau}\right)\right]. \tag{A10}$$

We have assumed here that k_o is real so that $k_o^2 > 0$. Equation (A10) remains valid if k_o is complex as long as the branch cut is defined so that k_o has a positive imaginary part. In one dimension with $R_1 = |x - x'|$ we have

$$\begin{aligned} G_1(x - x') &= \left[\frac{i}{2k_o}\right] e^{ik_o R_1} \\ &= \frac{1}{4\pi}\left[\frac{2\pi i R_1}{k_o}\right]^{1/2} \int_0^\infty \frac{d\tau}{\tau^{1/2}} \exp\left[\frac{ik_o R_1}{2}\left(\tau + \frac{1}{\tau}\right)\right]. \end{aligned} \tag{A11}$$

In two dimensions

$$\begin{aligned} G_2(\bar{x} - \bar{x}') &= \frac{i}{4} H_o^{(1)}(k_o R_2) \\ &= \frac{1}{4\pi} \int_0^\infty \frac{d\tau}{\tau} \exp\left[\frac{ik_o R_2}{2}\left(\tau + \frac{1}{\tau}\right)\right] \end{aligned} \tag{A12}$$

where

$$R_2 = \sqrt{(x - x')^2 + (y - y')^2},$$

and in three dimensions

$$\begin{aligned} G_3(\bar{x} - \bar{x}') &= \frac{1}{4\pi R_3} e^{ik_o R_3} \\ &= \frac{1}{4\pi}\left[\frac{k_o}{2\pi i R_3}\right]^{1/2} \int_0^\infty \frac{d\tau}{\tau^{3/2}} \exp\left[\frac{ik_o R_3}{2}\left(\tau + \frac{1}{\tau}\right)\right], \end{aligned} \tag{A13}$$

with

$$R_3 = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$

The integral representation Eq. (A10) may be used to derive the far-field expression for the Green's function G_n. If $k_o R_n \gg 1$, the integral over τ in Eq. (A10) can be evaluated by the method of stationary phase. The single, stationary-phase point is at $\tau = 1$, and we have, using Eq. (B4) of the following appendix,

$$G_n(\bar{x} - \bar{x}') \xrightarrow[k_o R_n \to \infty]{} \frac{1}{4\pi R_n}\left[\frac{k_o}{2\pi i R_n}\right]^{\frac{n-3}{2}} e^{ik_o R_n}. \tag{A14}$$

APPENDIX B
THE METHOD OF STATIONARY PHASE

Let

$$I(p) = \int_{\alpha}^{\beta} d\tau\, \psi(\tau) e^{ip\Theta(\tau)}, \tag{B1}$$

where Θ is a real function of τ with two continuous derivatives in (α, β) and p is some numerically large parameter. If $\Theta'(\tau_i) = 0$ $(i = 1, \ldots, m)$ with $\alpha < \tau_i < \beta$ and $\Theta''(\tau_i) \neq 0$, then*

$$I(p) = \sum_{i=1}^{m} \left[\frac{2\pi i}{p\Theta''(\tau_i)}\right]^{1/2} \psi(\tau_i) e^{ip\Theta(\tau_i)} + O\left(\frac{1}{p}\right). \tag{B2}$$

In the text we apply the method of stationary phase to integrals of the form

$$I_n = \int_0^{\infty} \frac{d\tau}{\tau^{n/2}} \exp\left[\frac{ik_o}{2}\left(a\tau + \frac{b}{\tau}\right)\right] \psi(\tau), \tag{B3}$$

where a and b are positive parameters and $k_o\sqrt{ab}$ is assumed large. We have

$$I_n \simeq \left(\frac{2\pi i}{k_o b}\right)^{1/2} \left(\frac{a}{b}\right)^{(n-3)/4} e^{ik_o\sqrt{ab}}\, \psi\left[\left(\frac{b}{a}\right)^{1/2}\right]. \tag{B4}$$

*C. Eckart, "The Approximate Solution of One-Dimensional Wave Equations," Rev. Mod. Phys. **20**, 399-417 (1948).